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A Hochschild homology criterium for the smoothness of an algebra

BACH (The Buenos Aires Cyclic Homology Group)[1]

0. Introduction

Let k be a field of characteristic zero. A k-algebra A is essentially of finite type (e.f.t.) if there exist $x_1, \ldots, x_n \in A$ and a multiplicative system $S \subseteq k[x_1, \ldots, x_n]$ such that $A = S^{-1}(k[x_1, \ldots, x_n])$. For example every finitely generated k-algebra is an e.f.t. algebra.

Rodicio states in [R] the following conjecture:

If A is a finitely generated k-algebra, and $HH_n(A) = 0$ for $n > n_0$, then A is a geometrically regular (i.e. smooth) k-algebra.

In [M-R] the authors prove this conjecture in the case where A is a locally complete intersection. Afterwards, [L-R] state more precisely that if the Hochschild homology of a locally complete intersection A is zero in an odd and in an even degree, then A is smooth.

In this work we deduce that this conjecture holds for A an arbitrary e.f.t. k-algebra from a more general result. More precisely:

0.1. THEOREM[2]. *Let k be a field of characteristic zero, $\bar{k}$ its algebraic closure, A an e.f.t. k-algebra and $p \in \operatorname{Spec}(\bar{k} \otimes_k A)$. If p is not regular, then $HH_i(A) \neq 0$ for every i congruent to $q = \max(j: \Omega^j(A_p) \neq 0)$ modulo 2.*

As an immediate consequence of the theorem we see that if A is an e.f.t. k-algebra, and $HH_i(A) = 0 = HH_{i'}(A)$ for some i odd and some i' even, then A is geometrically regular.

[1] The following people participated in this research: Antonio Campillo, Jorge Alberto Guccione, Juan José Guccione, María Julia Redondo, Andrea Solotar and Orlando E. Villamayor.

[2] When this paper had been submitted for publication, we received the manuscript [A-V], in which L. Avramov and M. Vigué-Poirrier proved this theorem in a more general setting, i.e., over a field of arbitrary characteristic.

This paper has two sections. In the first one we establish some preliminary results that will be used in the proof of Theorem 1. In the second one we prove the main result.

1. Preliminaries

In this section we recall some basic results from commutative algebra and establish some technical results that we shall need later.

1.1. *Commutative algebra*

1.1.1. THEOREM. ([K, Th. 7.14]). *Let A be an e.f.t. k-algebra and $p \in \operatorname{Spec}(A)$. Then the following facts are equivalent:*
(i) *A_p is geometrically regular.*
(ii) *$\Omega^1(A_p)$ is a free A_p-module of rank $\dim_p A$*

1.1.2. LEMMA. *Let k be an algebraically closed field. Every e.f.t. local k-algebra A is isomorphic to a quotient R/I of an e.f.t. local regular k-algebra (R, m) by an ideal $I \subseteq m^2$.*

Proof. It is clear that there exists an e.f.t. local regular k-algebra (R', m') and an ideal $I' \subseteq m'$ such that $A = R'/I'$. Let $\{f_1, \ldots, f_r\}$ be a regular sequence which is maximal amongst those included in $(I \cap m) \backslash m^2$. It is clear that $R = R'/\langle f_1, \ldots, f_r\rangle$ is an e.f.t. local regular k-algebra with maximal ideal $m = m'/\langle f_1, \ldots, f_r\rangle$.

1.2. *Hochschild homology*

1.2.1. DEFINITION. ([C-G-G]). A k-algebra R is called homologically regular. (It has the property $(\mathscr{R})$ in the terminology of [M-R]) if the map $\mu_*^R: (R \otimes \bar{R}^{\otimes^*}, b) \to (\Omega^*(R), 0)$ (see [L-Q]) is a quasi-isomorphism and $\Omega^1(R)$ is flat.

1.2.2. DEFINITION. A commutative differential graded k-algebra (k-CDGA) (R, d) is called homologically regular if $R = R_0 \otimes \wedge(V)$, with R_0 homologically regular and $\wedge(V)$ the free symmetric graded k-algebra generated by the graded k-vector space $V = V_1 \oplus V_2 \oplus \cdots$.

Given a graded k-vector space V we denote by $\bar{V}$ the graded k-vector space $\bar{V} = \bar{V}_2 \oplus \bar{V}_3 \oplus \cdots$, where $\bar{V}_j = V_{j-1}$ $(j \geq 1)$.

Let $(R_0 \otimes \wedge(V), d)$ be an homologically regular k-CDGA. We define $\beta: R_0 \otimes \wedge(V) \to \Omega^*(R_0) \otimes \wedge(V \oplus \bar{V})$ as the unique derivative of degree $+1$ which verifies:

(i) $\beta(a) = d_{DR}(a) \in \Omega^1(R_0)$, for $a \in R_0$,

(ii) $\beta(v) = \bar{v}$, for $v \in V_j$ $(j \geq 1)$

To each homologically regular k-DGA $(R_0 \otimes \wedge(V), d)$ we associate the commutative differential graded k-algebra $(\Omega^*(R_0) \otimes \wedge(V \oplus \bar{V}), \delta)$, where δ is the unique derivative of degree -1 which verifies

(i) $\delta_{|R_0 \otimes \wedge(V)} = d$,

(ii) $\delta_{|\Omega^*(R_0)} = 0$,

(iii) $\delta(\bar{v}) = -\beta(dv)$, $v \in V_j$ $(j \geq 1)$.

1.2.3. THEOREM. *The Hochschild homology $HH_*(R_0 \otimes \wedge(V), d)$ of an homologically regular algebra $(R_0 \otimes \wedge(V), d)$ is the homology of the k-CDGA $(\Omega^*(R_0) \otimes \wedge(V \oplus \bar{V}), \delta)$.*

Proof. See [C-G-G, Th. 2.1.5].

By definition, for $m \in \mathbb{N}$, $\wedge^\alpha(\bar{V})$ is the vector subspace of $\wedge(\bar{V})$ generated by the monomials $\bar{v}_{\gamma_1} \cdots \bar{v}_{\gamma_\alpha}$, with $\bar{v}_{\gamma_j} \in \bar{V}$. It is clear that

$$\delta\left(\bigoplus_{j=0}^{m} \Omega^j(R_0) \otimes \wedge(V) \otimes \wedge^{m-j}(\bar{V})\right) \subseteq \bigoplus_{j=0}^{m} \Omega^j(R_0) \otimes \wedge(V) \otimes \wedge^{m-j}(\bar{V})$$

Hence the Hochschild homology of $(R_0 \otimes \wedge(V), d)$ splits into the direct sum of the homologies of the complexes

$$\left(\bigoplus_{j=0}^{m} \Omega^j(R_0) \otimes \wedge(V) \otimes \wedge^{m-j}(V), \delta\right)$$

That is

$$HH_*(R_0 \otimes \wedge(V), d) = \bigoplus_{m \geq 0} H_*\left(\bigoplus_{j=0}^{m} \Omega^j(R_0) \otimes \wedge(V) \otimes \wedge^{m-j}(\bar{V}), \delta\right)$$

2. Proof of the theorem

We shall now prove the Theorem announced in the introduction.

Proof. Since $\Omega^i(A) \subseteq HH_i(A)\ \forall i \in \mathbb{N}$ it is clear that $HH_i(A) \neq 0\ \forall i \leq q$. Let's suppose now that $i = q + 2h$ and $h > 0$. Let $S = (\bar{k} \otimes_k A) \backslash p$. Since

$$S^{-1}(\bar{k} \otimes_k (HH_{q+2h}(A))) = HH_{q+2h}(S^{-1}(\bar{k} \otimes_k A)),$$

we can suppose that k is algebraically closed and A is a local e.f.t. k-algebra with maximal ideal p. By Lemma 1.1.2. A can be written as R/I, with (R, m) a local regular k-algebra and $I \subseteq m^2$. Let $\{P_1, \ldots, P_r\}$ be a set of generators of I. It can be easily seen that there is an homologically regular k-CDGA $(R \otimes \wedge(V), d)$ which verifies that

(i) $V_1 = kx_1 \oplus \cdots \oplus kx_r$, and V_j is finitely generated for $j > 1$

(ii) $d(x_j) = P_j\ (j = 1, \ldots, r)$

(iii) $H_j(R \otimes \wedge(V), d) = 0$, for all $j \geq 1$

Then the canonical projection $\pi\colon (R \otimes \wedge(V), d) \to A$ is a quasi-isomorphism. As a consequence,

$$HH_*(A) = HH_*(R \otimes \wedge(V), d) = \bigoplus_{m \geq 0} H_*\left(\bigoplus_{j=0}^{m} \Omega^j(R) \otimes \Lambda(V) \otimes \Lambda^{m-j}(\bar{V}), \delta\right).$$

Let $\{y_1, \ldots, y_s\}$ be a base of V_2. For every y_t there exist $F_{jt} \in R$ $(1 \leq j \leq r)$, such that $d(y_t) = \Sigma_{j=1}^{r} F_{jt} x_j$. It is easily seen that if $i = q + 2h$ with $h > 0$, then

(i) $\left(\bigoplus_{j=0}^{q+h} \Omega^j(R) \otimes \Lambda(V) \otimes \Lambda^{q+h-j}(\bar{V})\right)_\alpha = 0$, for $\alpha < i$,

(ii) $\left(\bigoplus_{j=0}^{q+h} \Omega^j(R) \otimes \Lambda(V) \otimes \Lambda^{q+h-j}(\bar{V})\right)_i = \bigoplus_{B(h)} \Omega^q(R)\bar{x}_1^{\alpha_1} \cdots \bar{x}_r^{\alpha_r}$

(iii) $\left(\bigoplus_{j=0}^{q+h} \Omega^j(R) \otimes \Lambda(V) \otimes \Lambda^{q+h-j}(\bar{V})\right)_{i+1} = \left(\bigoplus_{j=1}^{r} \bigoplus_{B(h)} \Omega^q(R)x_j \cdot \bar{x}_1^{\alpha_1} \cdots \bar{x}_r^{\alpha_r}\right)$

$\oplus \left(\bigoplus_{j=1}^{r} \bigoplus_{B(h-1)} \Omega^q(R)\bar{x}_1^{\alpha_1} \cdots \bar{x}_r^{\alpha_r} \cdot \bar{y}_t\right) \oplus \left(\bigoplus_{B(h+1)} \Omega^{q-1}(R)\bar{x}_1^{\alpha_1} \cdots \bar{x}_r^{\alpha_r}\right),$

where $B(i) = \{\alpha_1, \ldots, \alpha_r : \alpha_1 + \cdots + \alpha_r = i\}$.

Applying the definition of δ we have that, when $\omega_\alpha \in \Omega^\alpha(R)$:

(i) $\delta(\omega_q x_j \cdot \bar{x}_1^{\alpha_1} \cdots \bar{x}_r^{\alpha_r}) = (-1)^q \omega_q P_j \cdot \bar{x}_1^{\alpha_1} \cdots \bar{x}_r^{\alpha_r}$

(ii) $\delta(\omega_q \cdot \bar{x}_1^{\alpha_1} \cdots \bar{x}_r^{\alpha_r} \bar{y}_t) = (-1)^q \omega_q \left(\sum_{j=1}^{r} F_{jt}\bar{x}_j\right) \cdot \bar{x}_1^{\alpha_1} \cdots \bar{x}_r^{\alpha_r}$

(iii) $\delta(\omega_{q-1} \cdot \bar{x}_1^{\alpha_1} \cdots \bar{x}_r^{\alpha_r}) = (-1)^{q-1}\left(\sum_{j=1}^{r} \alpha_j \omega_{q-1} \wedge d_{DR}(P_j)\bar{x}_1^{\alpha_1} \cdots \bar{x}_j^{\alpha_j - 1} \cdots \bar{x}_r^{\alpha_r}\right).$

Let us now consider the map

$$\Theta\colon \bigoplus_{B(h)} \Omega^q(R)\bar{x}_1^{\alpha_1} \cdots \bar{x}_r^{\alpha_r} \to \Omega^q(R)$$

$$\omega_q \cdot \bar{x}_1^{\alpha_1} \cdots \bar{x}_r^{\alpha_r} \mapsto \omega_q P_1^{\alpha_1} \cdots P_r^{\alpha_r}$$

The map Θ has the following properties

(i) $\operatorname{Im}(\Theta) = \Omega^q(R)I^h$

(ii) $\Theta \circ \delta(\omega_q \cdot \bar{x}_1^{\alpha_1} \cdots \bar{x}_r^{\alpha_r} \bar{y}_t) = (-1)^q \omega_q \left(\sum_{j=1}^{r} F_{jt} P_j\right) \cdot P_1^{\alpha_1} \cdots P_r^{\alpha_r} = 0$

because $\sum_{j=1}^{r} F_{jt} P_j = d^2(y_t) = 0$

(iii) $\Theta \circ \delta(\omega_{q-1} \cdot \bar{x}_1^{\alpha_1} \cdots \bar{x}_r^{\alpha_r}) = (-1)^{q-1}\left(\sum_{j=1}^{r} \alpha_j \omega_{q-1} \wedge d_{DR}(P_j) P_1^{\alpha_1} \cdots P_j^{\alpha_j - 1} \cdots P_r^{\alpha_r}\right)$

which belongs to $\Omega^q(R)mI^h$, because $P_j \in m^2$ implies that $d_{DR}(P_j) \in m\Omega^1(R)$.

(iv) $\Theta \circ \delta(\omega_q x_j \cdot \bar{x}_1^{\alpha_1} \cdots \bar{x}_r^{\alpha_r}) = (-1)^q \omega_q P_j \cdot P_1^{\alpha_1} \cdots P_r^{\alpha_r}$,

which is an element of $\Omega^q(R)I^{h+1} \subseteq \Omega^q(R)mI^h$.

Then, $\Theta \circ \delta$ maps

$$\left(\bigoplus_{j=0}^{q+h} \Omega^j(R) \otimes \Lambda(V) \otimes \Lambda^{q+h-j}(\bar{V})\right)_{i+1}$$

into $\Omega^q(R)mI^h$. As a consequence, the Hochschild homology does not vanish at the desired degrees.

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Departamento de Matemática
Facultad de Ciencias Exactas y Naturales
Pabellón 1 – Ciudad Universitaria
(1428) Buenos Aires
Argentina
e-mail: vander@mate.edu.ar

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