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Hochschild cohomology algebra of abelian groups

By

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Abstract. In this paper we present a direct proof of what is suggested by Holm's results (T. Holm, The Hochschild cohomology ring of a modular group algebra: the commutative case, Comm. Algebra **24,** 1957–1969 (1996)): there is an isomorphism of algebras $HH^*(kG, kG) \to kG \otimes H^*(G, k)$ where G is a finite abelian group, k a ring, $HH^*(kG, kG)$ is the Hochschild cohomology algebra and $H^*(G, k)$ the usual cohomology algebra.

This result agrees with the well-known additive structure result in force for any group G; we remark that the multiplicative structure result we have obtained is quite similar to the description of the monoidal category of Hopf bimodules over kG given in "C. Cibils, Tensor product of Hopf bimodules, to appear in Proc. Amer. Math. Soc.". This similarity leads to conjecture the structure of $HH^*(kG, kG)$ for G a finite group.

Introduction. The multiplicative structure of the Hochschild cohomology algebra of an abelian group algebra over a field of finite characteristic has recently been obtained by Holm [7], using computations based on a paper of the Buenos Aires Cyclic Homology Group [1].

The purpose of this note is to present a direct and easy proof of what is suggested by Holm's result, namely that there is an isomorphism of algebras

$$HH^*(kG, kG) = kG \otimes H^*(G, k)$$

where G is a finite abelian group, k any commutative ring, kG the group algebra, $HH^*(kG, kG)$ the Hochschild cohomology algebra of kG with coefficients in the kG-bimodule given by the algebra and $H^*(G, k)$ the usual group cohomology algebra with coefficients in the trivial module k. This multiplicative description of $HH^*(kG, kG)$ agrees with the well-known additive result in force for any group G:

$$HH^*(kG, kG) = \prod_{c \in \mathscr{C}} H^*(Z_c, k)$$

where $\mathscr{G}$ is the set of conjugacy classes of G and Z_c is the centralizer of an element of C, see [3], [2] or [9]. The remark in [2] concerning the multiplicative structure besides the additive decomposition of the Hochschild cohomology does not provide any result. Actually the problem for a non abelian group is a difficult task. It is interesting to notice that the behavior

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of Hopf bimodules over a group algebra considered in [4] is quite parallel to the structure of the Hochschild cohomology algebra for finite abelian groups. Indeed the results of [4] restricted to the abelian case show that the category of Hopf bimodules over an abelian finite group G is isomorphic to the graded category $[\mathrm{mod}\, kG]G$, as monoidal categories. This result is analogous to the one obtained in the present paper, namely that the Hochschild cohomology algebra of kG is isomorphic to the algebra $kG \otimes H^*(G,k)$. Following this parallelism observed for abelian groups, leads to conjecture the following for any finite group, as in [4].

Let $M(kG) = \bigoplus_{C\in\mathscr{C}} H^*(Z_{u(C)}, k)C$, namely the k-module of formal linear combinations of conjugacy classes with coefficients in the respective cohomology group, where u is a choice of an element in each conjugacy class. Let mA and nB be elementary elements in $M(kG)$, with $m \in H^*(Z_a,k)$ and $n \in H^*(Z_b,k)$. The product $mA\cdot nB = \sum_{C\in\mathscr{C}} x^C_{A,B}C$ is defined by

$$x^C_{A,B} = \sum_{E^C_{A,B}/Z_c} \mathrm{Ind}^{Z_c}_{Z^K_a\cap Z^L_b}\left(\mathrm{Res}^{Z^K_a}_{Z^K_a\cap Z^L_b}\, m^K \smile \mathrm{Res}^{Z^L_b}_{Z^K_a\cap Z^L_b}\, n^L\right)$$

where m^K and n^L denote elements corresponding to m and n in $H^*(Z^K_a,k)$ and $H^*(Z^L_b,k)$ obtained through conjugation, Ind and Res are the induction and restriction maps defined in cohomology, and $\smile$ is the cup product in the corresponding cohomology algebra.

Conjecture. The Hochschild cohomology algebra $HH^*(kG,kG)$ is isomorphic to $M(kG)$.

Notice that cohomological results for a field k are interesting only if the characteristic divides the order of G – otherwise the statement is essentially about the null vector space –, while the monoidal equivalence quoted above is interesting at any characteristic.

It is known from [8] that Hopf bimodules over a Hopf algebra H are equivalent to left modules over the quantum double $\mathscr{D}(H)$, and from [6] that they correspond exactly to modules over an explicit associative algebra X. It would be interesting to understand a relation between the Hochschild cohomology algebra of the algebras $\mathscr{D}(H)$ or X and the monoidal structure of $\mathrm{mod}\,\mathscr{D}(H)$ or $\mathrm{mod}\,X$.

We have chosen to present this results for a *finite* abelian group in its cohomological version. Alternatively we could as well consider any abelian group and obtain an equality of coalgebras

$$HH_*(kG,kG) = k^G \otimes H_*(G,k)$$

where k^G is the coalgebra of all the k-valued functions over G. This of course implies the cohomological result for G a finite abelian group and k a field, since for a finite group we have firstly an isomorphism of algebras

$$(HH_*(kG,kG))' = HH^*(kG,(kG)')$$

where V' denotes the dual of a vector space V, and secondly finite dimensional group algebras have the property that the kG-bimodule kG is self-dual, using the map which sends an element s of G to the Dirac mass $\delta_{s^{-1}}$.

2. Cohomology algebras. Let G be a finite abelian group and k a commutative ring.

Theorem 2.1. *There is an isomorphism of graded rings*

$$HH^*(kG,kG) \to kG \otimes H^*(G,k).$$

We recall first the definition of both rings (see for instance [3]). The Hochschild cohomology $HH^*(\Lambda,M)$ of a k-algebra Λ with coefficients in a Λ-bimodule M is the cohomology of the cochain complex:

$$(\mathscr{H}): 0 \to M \xrightarrow{d_0} \mathrm{Hom}_k(\Lambda,M) \xrightarrow{d_1} \cdots \to \mathrm{Hom}_k(\Lambda^{\otimes i},M) \xrightarrow{d_i} \cdots$$

where differentials are given by:

$$d_0(m)(x) = xm - mx$$

and

$$(d_i f)(x_1\otimes\cdots\otimes x_{i+1}) = x_1 f(x_2\otimes\cdots\otimes x_{i+1})$$
$$+\sum(-1)^j f(x_1\otimes\cdots\otimes x_j x_{j+1}\otimes\cdots\otimes x_{i+1}) + (-1)^{i+1} f(x_1\otimes\cdots\otimes x_i)x_{i+1}.$$

We provide the definition of the Hochschild homology for a possible use according to the remark at the end of the Introduction; $HH_*(\Lambda,M)$ is the homology of the chain complex

$$\cdots \to \Lambda^{\otimes i}\otimes M \xrightarrow{\delta_{i-1}} \cdots \xrightarrow{\delta_1} \Lambda\otimes M \xrightarrow{\delta_0} M \to 0$$

where the differentials are given by:

$$\delta_0(x\otimes m) = xm - mx$$

and

$$\delta_{i-1}(x_1\otimes\cdots\otimes x_i\otimes m) = (x_2\otimes\cdots\otimes x_i\otimes m x_1) +$$
$$+\sum(-1)^j(x_1\otimes\cdots\otimes x_j x_{j+1}\otimes\cdots\otimes x_i\otimes m) + (-1)^i(x_1\otimes\cdots\otimes x_{i-1}\otimes x_i m).$$

It follows immediately that if k is a field and M is finite dimensional as vector space, there is a canonical isomorphism

$$(HH^*(\Lambda,M))' = HH_*(\Lambda,M').$$

In case $M=\Lambda$ considered as a Λ-bimodule with left and right actions given by multiplication, the Hochschild cohomology $HH^*(\Lambda,\Lambda)$ becomes a ring through the cup product

$$(f\smile g)(x_1\otimes\cdots\otimes x_i\otimes x_{i+1}\otimes\cdots\otimes x_{i+j}) = f(x_1\otimes\cdots\otimes x_i)g(x_{i+1}\otimes\cdots\otimes x_{i+j})$$

using the product of Λ. This provides the cochain algebra with a structure of differential graded algebra. Through the canonical isomorphism above, we obtain a coalgebra structure on the Hochschild homology.

The usual cohomology $H^*(G,k)$ of a group G can be defined as the Hochschild cohomology $HH^*(kG,k)$ with coefficients in the trivial bimodule k. Since $\mathrm{Hom}_k(kG^{\otimes i},M) = \mathrm{Map}(G^{\times i},M)$, the cohomology $H^*(G,k)$ is the cohomology of the cochain complex:

$$0 \to k \xrightarrow{d_0} \mathrm{Map}(G,k) \to \cdots \to \mathrm{Map}(G^{\times i},k) \xrightarrow{d_i} \cdots$$

where

$$d_0(1)(s) = s1 - 1s = 0$$

$$(d_i f)(s_1, \dots, s_{i+1}) = f(s_2, \dots, s_{i+1})$$
$$+ \sum (-1)^j f(s_1, \dots, s_j s_{j+1}, \dots, s_{i+1}) + (-1)^{i+1} f(s_1, \dots, s_i).$$

The ring structure is provided in a similar way as before:

$$(f \smile g)(s_1, \dots, s_{i+j}) = f(s_1, \dots, s_i) g(s_{i+1}, \dots, s_{i+j})$$

using now the product of k.

Proof of the theorem. We consider first an arbitrary finite group G. For each conjugacy class C, define:

$$\mathscr{H}_i^C = \{f \mid f(s_1, \dots, s_i) \in k[s_1 \cdots s_i C] \text{ for all } s_1, \dots, s_i \in G\}$$

where $s_1 \cdots s_i C$ denotes the conjugacy class of C translated by $s_1 \cdots s_i$, and $k[s_1 \cdots s_i C]$ is the k-submodule of kG generated by this set. Let $\mathscr{H}^C = \bigoplus_i \mathscr{H}_i^C$. Actually $\mathscr{H}^C$ is a subcomplex of $\mathscr{H}$. Indeed, let f be a cochain of $\mathscr{H}_i^C$ and consider the above formula providing the differential. In order to verify that each summand of df evaluated at $(s_1, \dots, s_{i+1})$ belongs to $k[s_1 \cdots s_{i+1} C]$ notice that only the last one needs attention: $f(s_1, \dots, s_i) s_{i+1}$ is in $k[s_1 \cdots s_i C s_{i+1}]$ but since C is a conjugacy class, we have

$$s_1 \cdots s_i C s_{i+1} = s_1 \cdots s_i s_{i+1} s_{i+1}^{-1} C s_{i+1} = s_1 \cdots s_{i+1} C.$$

We assert now that $\mathscr{H} = \bigoplus_{C \in \mathscr{C}} \mathscr{H}^C$ where $\mathscr{C}$ is the set of all conjugacy classes of G. Let f be an i-cochain and consider for each element $s \in G$ the partition $G = \coprod_{C \in \mathscr{C}} sC$, and the corresponding canonical projections $\pi_s^C : kG \to k[sC]$. Define f^C by $f^C(s_1, \dots, s_i) = \pi_{s_1 \dots s_i}^C f(s_1, \dots, s_i)$, which is clearly a cochain in $\mathscr{H}_i^C$ for each conjugacy class. Moreover we have $f = \sum_{C \in \mathscr{C}} f^C$ since this equality is verified for each element of $G^{\times i}$.

Assume now $f = \sum_{C \in \mathscr{C}} f^C = 0$ for some set of cochains $\{f^C\}_{C \in \mathscr{C}}$ with $f^C \in \mathscr{H}_i^C$. Since for each $(s_1, \dots, s_i)$ we have a direct sum decomposition $kG = \bigoplus_{C \in \mathscr{C}} k[s_1 \cdots s_i C]$, we infer that $f^C(s_1, \dots, s_i)$ is zero and $f^C = 0$.

For G an abelian group, conjugacy classes are elements of G, hence a cochain of $f^c \in \mathscr{H}_i^c$ for $c \in G$ attributes a scalar multiple of $s_1 \dots s_i c$ for each element $(s_1, \dots, s_i) \in G^{\times i}$; we denote the corresponding scalar $\overline{f^c}(s_1, \dots, s_i)$ and notice that we obtain this way a map $\overline{f^c} : G^{\times i} \to k$. To the cochain f^c we associate $\varphi(f^c) = \overline{f^c} \otimes c$, a cochain of the complex computing the usual cohomology of G tensored by the group ring. There is no difficulty to prove that this map provides an isomorphism of cochain complexes. In order to verify its compatibility with respect to the products, notice first that if $f^c \in \mathscr{H}_i^c$ and $g^d \in \mathscr{H}_j^d$, we have that $f^c \smile g^d$ is a cochain of $\mathscr{H}_{i+j}^{cd}$. Moreover, the scalar elements are related as follows:

$$\lambda_{(s_1, \dots, s_i, t_1, \dots, t_j)}(f^c \smile g^d) = \lambda_{(s_1, \dots, s_i)}(f^c) \lambda_{(t_1, \dots, t_j)}(g^d).$$

We obtain finally that $\varphi(f^c \smile g^d) = \varphi(f^c)\varphi(g^d)$ as required.

Concerning the non-abelian case, some facts which agree with the conjecture quoted at the introduction can be already derived from the previous proof. The direct summands corresponding to elements of the center Z of G provide a subalgebra of $HH^*(kG, kG)$ isomorphic to $kG \otimes H^*(Z, k)$; moreover, the orbits of the conjugacy classes under the action of G produce direct summands which are bimodules over the subalgebra mentioned above.

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