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# BIVARIANT COHOMOLOGY AND $S^1$-SPACES

BY

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RÉSUMÉ. — Le but de cet article est d'étendre au cadre bivariant le théorème de Jones, Goodwillie et Burghelea-Fiedorowicz (cf. [J], [G], [B-F]), qui prouve l'isomorphisme entre la cohomologie cyclique du complexe singulier d'un $S^1$-espace $X$ et la cohomologie $S^1$-équivariante de $X$. Nous faisons également la comparaison entre la longue suite exacte de Connes (théorie cyclique) et la longue suite exacte de Gysin (théorie $S^1$-équivariante).

Nous prouvons aussi que dans quelques cas, la cohomologie cyclique bivariante peut être calculée comme la cohomologie cyclique (monovariante) d'un certain complexe mixte.

ABSTRACT. — The purpose of the following work is to provide a generalization to the bivariant setting of the theorem of Jones, Goodwillie and Burghelea-Fiedorowicz (cf. [J], [G], [B-F]), which proves the existence of an isomorphism between the cyclic cohomology of the singular complex module of an $S^1$-space $X$ and the $S^1$-equivariant cohomology of $X$. We also compare Connes' long exact sequence in the cyclic theory with Gysin's long exact sequence in the $S^1$-equivariant theory.

We see that in some cases the bivariant cyclic cohomology can be computed as the (monovariant) cyclic cohomology of a mixed complexe.

## 0. Introduction

The bivariant version of cyclic cohomology was introduced by JONES and KASSEL in [J-K]. In the other hand, there is a topological definition for $S^1$-spaces $X$ and $Y$ of the bivariant $S^1$-equivariant cohomology, denoted $H^*_{S^1}(X,Y)$ which can be found in [C].

In the following work we prove the bivariant version of the theorem of JONES [J], GOODWILLIE [G] and BURGHELEA-FIEDOROWICZ [B-F], which

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says that if $X$ is an $S^1$-space, then its equivariant cohomology is isomorphic to the cyclic cohomology of the singular complex module of $X$ :

$$H^*_{S^1}(X) = HC^*\big(\mathbb{S}.(X)\big).$$

One of the main results is the following :

THEOREM. — *Let $X$ and $Y$ have the homotopy type of $CW$-complexes equipped with a pointed $S^1$-action, such that $Y$ has the homotopy type of a finite complex. Then there exists a natural isomorphism :*

$$H^n_{S^1}(X,Y) \cong \overline{H}C^n\big(\mathbb{S}.X, \mathbb{S}.Y\big)$$

*where $\overline{H}C^n(\mathbb{S}.X, \mathbb{S}.Y)$ is the reduced bivariant cyclic cohomology of $\mathbb{S}.X$ and $\mathbb{S}.Y$.*

This isomorphism sends Connes' long exact sequence in bivariant cyclic cohomology to a Gysin long exact sequence of $X$ and $Y$ in the topological context.

We also prove that in certain cases the bivariant cyclic cohomology of two cyclic modules $M.$ et $N.$ can be computed as the cyclic cohomology of the mixed complex $M.\otimes DN.$, where $DN.$ is the dual of $N.$, considered as a chain complex.

The paper is organized as follows :

In sections 1–3 we recall the definitions and some properties of bivariant cohomology, bivariant cyclic cohomology and the stable homotopy category Stab, respectively. This category, studied principally in [D-P], is used to provide an intermediate result during the proof.

In section 4 we extend the theorem of [J], [G] and [B-F] to the Stab category (PROPOSITION 4.1).

Section 5 gives the following preliminary result which is used in the proof of the main theorem :

PROPOSITION. — *Under the hypotheses of the theorem, there is a natural isomorphism between $HC^n(\mathcal{S}.X, \mathcal{S}.Y)$ and $HC^n(\mathcal{S}.(X\wedge DY))$, where $DY$ is the Spanier-Whitehead dual of $Y$ in the* Stab *category and $\mathcal{S}.$ denotes the reduced singular complex module.*

This result is proved in section 6.

Finally, in section 7, we show that the diagram which relates the reduced Gysin long exact sequence of $ES^1\times_{S^1}(X\wedge DY)$ and Connes' long exact sequence of the reduced bivariant cyclic cohomology of $\mathbb{S}.X$ and $\mathbb{S}.Y$ is commutative.





All the spaces that we are going to consider have the homotopy type of a CW-complex, are connected and base pointed.

## 1. Bivariant cohomology

Given two $CW$-complexes $X$ and $Y$, their bivariant cohomology with integral coefficients is defined, using maps of spectra as $[\Sigma^\infty X, \Sigma^\infty Y\wedge \mathbf{H}]$, where $\mathbf{H}$ is the Eilenberg-Mac Lane spectrum, $\mathbf{H}_i = K(\mathbb{Z}, i)$, $\Sigma^\infty X$ is the spectrum defined by $(\Sigma^\infty X)_i = S^i\wedge X$ and $[\ ,\ ]$ denotes homotopy classes of morphisms that fix the base point [C, p. 3].

As $(\Sigma^\infty Y\wedge \mathbf{H})_n = Y\wedge \mathbf{H}_n$ [S, Cor. 13.39], we may define :

$$H^i(X,Y) = \varinjlim_j [\Sigma^j X, Y\wedge K(\mathbf{Z}, j+i)] \qquad (i\in\mathbb{Z}).$$

There are other definitions of the same object which are equivalent, such as : $H^i(X,Y)$ is the group of chain homotopy classes of chain maps of degree $i$ from the reduced singular chain complex of $X$ to the reduced singular chain complex of $Y$ [C-S, p. 398]), and one has a split short exact sequence

$$\begin{aligned} 0\to \operatorname{Ext}^{n+1}\big(\overline{H}_*(X), \overline{H}_*(Y)\big) &\longrightarrow H^n(X,Y) \\ &\longrightarrow \operatorname{Hom}_{-n}\big(\overline{H}_*(X), \overline{H}_*(Y)\big)\to 0, \end{aligned}$$

where $\overline{H}$ denotes reduced homology : $\overline{H}(X) = H(X)/H(^*)$.

As a consequence, $H^n(X,Y)$ is $H_{-n}\big(\operatorname{Hom}(\mathcal{S}.X, \mathcal{S}.Y)\big)$.

## 2. Bivariant cyclic cohomology

For the definition and properties of cyclic $k$-modules (where $k$ is a commutative ring with unit), we refer to [Co] , [L1] and [L2].

We recall that a mixed complex $(M,b,B)$ is a nonnegatively graded $k$-module $(M_n)_{n\in\mathbb{N}}$ endowed with a degree $-1$ morphism $b$ and a degree $+1$ morphism $B$, such that $b^2 = B^2 = [B,b] = 0$.

The cyclic homology of a cyclic $k$-module is defined in [Co] and [L2], and the cyclic homology of a mixed complex is defined in [B] and [K1].

Given cyclic $k$-modules $M.$ and $N.$, KASSEL [K2] has defined the bivariant Hochschild cohomology of $M.$ and $N.$ as follows :

*Definition.* — $HH^n(M,N) = H_{-n}(\mathrm{Hom}(M.,N.),\mathrm{d})$ $(n \in \mathbb{Z})$, where $\mathrm{d}(f) = b_N \cdot f - (-1)^{\deg(f)} f \cdot b_M$ and $\mathrm{Hom}(M.,N.)$ is the complex such that $(\mathrm{Hom}(M.,N.))_j = \Pi_p \mathrm{Hom}(M_p, N_{p-j})$.

Given augmented cyclic $k$-modules $M.$ and $N.$, their reduced bivariant Hochschild cohomology $\overline{HH}^n(M.,N.)$ is defined as $HH^n(\mathbb{M}.,\mathbb{N}.)$, where $M. = k \oplus \mathbb{M}.$, $N = k \oplus \mathbb{N}.$. It verifies that :

$$HH^n(M.,N.) = \overline{HH}^n(M.,N.) \oplus HH^n(k) \oplus \overline{HH}^n(M.) \oplus HH^n(k.,\mathbb{N}.).$$

So we observe that $H^n(X,Y) \cong \overline{HH}^n(\mathbb{S}.X,\mathbb{S}.Y)$.

Jones and Kassel [J-K] have also defined the bivariant cyclic cohomology of $M.$ and $N.$ in the following way : to the mixed complex $(M,b,B)$ is associated the complex

$$\beta(M) = k[u] \underline{\otimes} M,$$

(where $\deg(u) = 2$), with differential

$$\mathrm{d}(u^n \otimes m) = \begin{cases} u^n \otimes bm + u^{n-1} \otimes Bm & \text{if } n > 0, \\ u^n \otimes bm & \text{if } n = 0. \end{cases}$$

The natural projection $S : \beta(M) \to \beta(M)[2]$ is given by :

$$S(u^n \otimes m) = \begin{cases} u^{n-1} \otimes m & \text{if } n > 0, \\ 0 & \text{if } n = 0, \end{cases}$$

which is a morphism of complexes.

The module $\beta(M)$ is then called an *S-module.*

We consider $\mathrm{Hom}_S(\beta(M),\beta(N))$, the submodule of $\mathrm{Hom}(\beta(M),\beta(N))$ consisting of elements which commute with $S$.

*Definition.* — $HC^n(M,N) = H_{-n}(\mathrm{Hom}_S(\beta(M),\beta(N))$ $(n \in \mathbb{Z})$.

*Remarks* :

1) $HC^n$ is a contravariant functor in $M$ and a covariant functor in $N$.
2) If $N = k$, $HC^i(M,k) = HC^i(M)$.
3) If $M = k$, $HC^i(k,N) = HC^-_{-i}(N)$.

(For a definition of $HC^-_*$, see [J].)

Following the ideas of [L-Q, § 4], the following definition of the reduced bivariant cyclic cohomology is given in [K3, 8.2].





*Definition.*—If $M.$ and $N.$ are augmented cyclic $k$-modules, the reduced bivariant cyclic cohomology of $M.$ and $N.$ is $\overline{HC}^n(M.,N.) = HC^n(\mathbb{M}.,\mathbb{N}.)$, where $M. = k. \oplus \mathbb{M}.$ and $N. = k. \oplus \mathbb{N}.$.

*Remark :*

$$HC^n(M.,N.) = \overline{HC}^n(M.,N.) \oplus \overline{HC}^n(M.) \oplus \overline{HC}^-_{-n}(N.) \oplus HC^n(k.).$$

*Examples.* — If $X$ and $Y$ are $S^1$-spaces, then their reduced singular complex $k$-modules, denoted $\mathcal{S}.X$ and $\mathcal{S}.Y$ are generated by their reduced singular complexes $\mathcal{S}.X$ and $\mathcal{S}.Y$. These $k$-modules are not only simplicial $k$-modules but also cyclic $k$-modules, with the cyclic action defined by (see [G]) :

$$\begin{array}{ccc} C_n \times \mathcal{S}_n(X) & \longrightarrow & \mathcal{S}_n(X) \\ (t_n, s) & \mapsto & t_n \cdot \sigma, \end{array}$$

where $t_n \cdot \sigma(u_0,...,u_n) = \mathrm{e}^{2\pi i u_0} \cdot \sigma(u_1,..,u_n,u_0)$.

In this case $S.X = k \oplus \mathcal{S}.X$ and the same for $Y$, so the $S^1$-spaces $X$ and $Y$ give rise to the bivariant cyclic cohomology groups $HC^n(\mathcal{S}.X,\mathcal{S}.Y)$ $(n \in \mathbb{Z})$, which by definition are the reduced bivariant cyclic cohomology groups of $\mathbb{S}.X$ and $\mathbb{S}.Y$, $\overline{HC}^n(\mathbb{S}.X,\mathbb{S}.Y)$.

## 3. The stable homotopy category

From now on all the spaces considered are base pointed, compactly generated $CW$-complexes.

**3.1.** — We recall from [D-P] that the stable homotopy category Stab is the category whose objects are pairs $(X,n)$, where $X$ is a space, $n \in \mathbb{Z}$, and whose maps are :

$$\mathrm{Stab}((X,n);(Y,m)) = \varinjlim_k [\Sigma^{n+k}X, \Sigma^{m+k}Y].$$

The product $(X,n) \otimes (Y,m) = (X \wedge Y, n+m)$ makes Stab a monoidal category.

We shall make use of the following objects :

*Definitions :*

(1) Given $(X,n)$, if $(X',n')$ is an object in Stab such that

$$\mathrm{Stab}((X,n) \otimes (Z,k),(S^0,0))$$

is isomorphic to $\mathrm{Stab}\big((Z,k);(X',n')\otimes(S^0,0)\big)$, for every object $(Z,k)$ in Stab, then $(X',n')$ is called (up to canonic isomorphism) the *weak dual* of $(X,n)$ and denoted $D(X,n)$.

(2) If $\mathrm{Stab}\big((X,n)\otimes((Z,k),(W,\ell)\big)$ is isomorphic to

$$\mathrm{Stab}\big((Z,k),(X',n')\otimes(W,\ell)\big)$$

for every pair of objects $(Z,k)$ and $(W,\ell)$ in Stab, then $(X',n')$ is called (up to canonic isomorphism) the *strong dual* of $(X,n)$ and also denoted $D(X,n)$.

The uniqueness of a (weak) dual object is assured by [S, cor. 14.25] and the existence of a strong dual for a finite $CW$-complex, by [S, th.14.34].

Then $D:\mathrm{Stab}^*\to\mathrm{Stab}$ is a contravariant functor, where $\mathrm{Stab}^*$ is the full subcategory whose objects are those $(X,n)$ such that $X$ is a finite $CW$-complex.

According to the definition of the homology of spectra [S], it is clear that :

$$H_n\big((X,h)\big)=H_{n-h}(X).$$

Similarly, for cohomology we have that $H^n\big((X,h)\big)=H^{n-h}(X)$.

## 4. $S^1$-spaces

Let $Z$ be an $S^1$-space, such that the action of $S^1$ over $Z$ is pointed, that is, it preserves the base point (ex. : $Z=\mathrm{Map}(S^1,Z')$), and consider, for $n\in\mathbb{Z}$, the $S^1$-space $\Sigma^nZ=S^n\wedge Z$ with the trivial action of $S^1$ on $S^n$.

In this situation we want to define $ES^1\times_{S^1}(Z,n)$, for $(Z,n)\in\mathrm{Stab}$. We observe that if $n=0$, then

$$ES^1\times_{S^1}(Z,0)=ES^1\times_{S^1}Z=(ES^1\times_{S^1}Z,0).$$

If $n>0$, we identify $ES^1$ with $(ES^1,0)$, then

$$ES^1\times_{S^1}(Z,n)=ES^1\times_{S^1}(\Sigma^nZ,0)=(ES^1\times_{S^1}\Sigma^nZ,0).$$

As the space $ES^1\times_{S^1}\Sigma^nZ$ is homotopy equivalent to $\Sigma^n(ES^1\times_{S^1}Z)$, we define $ES^1\times_{S^1}(Z,n)=(ES^1\times_{S^1}Z,n)$ and we want to show now that, as in the case of $S^1$-spaces, we have :

PROPOSITION 4.1. — $H_*\big((ES^1\times_{S^1}Z,n)\big)=HC_*\big(\mathbb{S}.(Z,n)\big)$.

*Proof.* — We first observe that $\mathbb{S}.(Z,n)$ though not a cyclic module is a mixed complex.





We have already seen that :

$$H_*\big((ES^1\times_{S^1}Z,n)\big)=H_{*-n}(ES^1\times_{S^1}Z)$$

and by [J], [G] and [B-F], this last term equals $HC_{*-n}(\mathbb{S}.Z)$.

In order to calculate $HC_*\big(\mathbb{S}.(Z,n)\big)$, we use the $\beta$-complex of [L-Q], whose total complex is such that $(\mathrm{Tot}\,\beta)i=(\mathrm{Tot}\,\beta')_{i-n}$ (where $\beta'$ is the double complex of $\mathbb{S}.Z$). We obtain that $HC_*(\mathbb{S}.(Z,n))$ is isomorphic to $HC_{*-n}(\mathbb{S}.Z)$.

COROLLARY 4.2. — *$\overline{H}C^*\big(\mathcal{S}.(Z,n)\big)$ is isomorphic to $\overline{H}^*_{S^1}((Z,n))$, where $\overline{H}^*_{S^1}$ is the $BS^1$ reduced cohomology (i.e. $\overline{H}^n_{S^1}(\mathrm{pt})=\overline{H}^n(BS^1)=0$ for all $n$).*

## 5. Duality, bivariant Hochschild cohomology and bivariant cyclic cohomology

Let us consider now the category $\mathrm{Ho}(\partial\text{-mod}_k)$, whose objects and morphisms are respectively chain $k$-complexes and homotopy classes of chain maps.

The tensor product of complexes makes $\mathrm{Ho}(\partial\text{-mod}_k)$ a monoidal category, with neutral object $I=(I_q)_{q\in\mathbb{Z}}$ (where $I_q=k$ if $q=0$ and $0$ if not).

Every chain complex $A$ has a weak dual $DA$ defined by $(DA)_q=\mathrm{Hom}_k(A_{-q},k)$ and, by [D-P], a chain complex $A$ is strongly dualizable in $\mathrm{Ho}(\partial\text{-mod}_k)$ if and only if it has the homotopy type of a finitely generated and projective chain complex.

DOLD and PUPPE give an extension $S'$ to the category Stab of the functor which associates to a pointed space $(X,x_0)$, its singular complex module $S.X$, such that

$$H_n\big(\mathbb{S}'\cdot(X,k)\big)=\begin{cases}H_{n-k}(\mathbb{S}.X) & \text{if } n-k\geq 0,\\ 0 & \text{if } n-k<0.\end{cases}$$

*Remark.* — Singular reduced homology with coefficients in $k$ in the Stab category coincides with the definition given in this category by means of spectra, and similarly for cohomology.

In this context we have the following result :

PROPOSITION 5.1. — *If $M.$ and $N.$ are simplicial modules, there is an isomorphism :*

$$HH^*(M.,DN)\cong HH^*(M\otimes N.).$$

*Proof.*

$$\begin{aligned} HH^*(M., DN.) &= H_{-*}(\mathrm{Hom}(M., DN.)) \\ &= H_{-*}(\mathrm{Hom}(M. \otimes N., k)) \\ &= HH^*(M. \otimes N.). \end{aligned}$$

The second equality is obtained by definition of the weak dual in the category.

Similarly, we can prove the following results :

COROLLARY 5.2.

(i) *If $N.$ is such that $DDN.$ is homotopically equivalent to $N.$ (for example, is $N.$ is finitely generated and projective), then :*

$$HH^*(M., N.) \quad \textit{is isomorphic to} \quad HH^*(M. \otimes DN).$$

(ii) *If $M.$ is such that $DDM.$ is homotopically equivalent to $M.$ (for example, is $M.$ is finitely generated and projective), then :*

$$HH^*(M., N.) \quad \textit{is isomorphic to} \quad HH_{-*}(DM. \otimes N.).$$

We have already seen that $\overline{H}_*^{\mathcal{S}^1}(X \wedge DY) = HC_*(\mathcal{S}.(X \wedge DY))$ and the same for cohomology.

Now we want to show that $HC^n(\mathcal{S}.(X \wedge DY))$ is isomorphic to $\overline{HC}^n(\mathbb{S}.X, \mathbb{S}.Y)$. We shall consider a more general framework.

Let $C$ be the subcategory of $\mathrm{Ho}(\partial\text{-mod}_k)$ whose objects are the chain complexes which have a degree $-2$ action $S$ and whose morphisms are those ones of $\mathrm{Ho}(\partial\text{-mod}_k)$ that commute with $S$.

The subcategory $C$ consists, then, of the complexes which are $k[u]$-comodules ($\mathrm{dg}(u) = 2$), where $k[u]$ is a coalgebra wih coproduct

$$\triangle(u^n) = \sum_{i+j=n} u^i \otimes u^j$$

and counit $\kappa(u^i) = 1$ if $i = 0$ and $0$ if not.

The cotensor product $\square_{k[u]}$ makes $C$ a monoidal category.

PROPOSITION 5.4. — *Let $(M, b, B)$ be a mixed complex and let $DM.$ denote the weak dual of $M.$ in $\mathrm{Ho}(\partial\text{-mod}_k)$, and ${}_BM = k[u] \underline{\otimes} M.$ the associated total complex, with differential*

$$\partial(u^i \underline{\otimes} m) = u^i \underline{\otimes} bm + u^{i-1} \underline{\otimes} Bm.$$

Then $DM.$ is also a mixed complex and ${}_BDM$ is the weak dual of ${}_BM$ in $C$. Moreover, if $M.$ has the homotopy type of a projective finitely generated chain complex (and so $DM.$ is the strong dual of $M.$ in $\mathrm{Ho}(\partial\text{-mod}_k)$), then ${}_BDM$ is the strong dual of ${}_BM$ in $C$.





*Proof.* — We have the mixed complex $(M, b, B)$. Then $DM.$ is the complex defined by $(DM)_j = \mathrm{Hom}_k(M_{-j}, k)$ ($DM.$ is zero in positive degrees).

We define $b : DM_j \to DM_{j-1}$ and $B : DM_j \to DM_{j+1}$ from $b$ and $B$ by composition. Then $(DM., b, B)$ is also a mixed complex.

The fact that the evaluation $\varepsilon : DM. \otimes M. \to .k$ is a morphism of mixed complexes implies that ${}_BDM$ is the dual of ${}_BM$ in $C$.

If $DM.$ is the strong dual of $M.$ in $\mathrm{Ho}(\partial\text{-mod}_k)$, we consider the morphisms of $k[u]$-comodules :

$$\varepsilon' : {}_BM \,\square_{k[u]}\, {}_BDM \to k[u] \quad \text{(evaluation) and}$$

$$\nu' : k[u] \to {}_BM \,\square_{k[u]}\, {}_BDM \quad \text{(coevaluation).}$$

The last one is defined by $\nu'(u^j) = \sum_i (u^j \underline{\otimes} a_i) \otimes f_i$ (where $\{a_1, ..., a_n\}$ is a basis of $\bigcup_{q \in \mathbb{Z}} M_q$ and $\{f_1, ..., f_n\}$ is the dual basis in $\bigcup_{q \in \mathbb{Z}} (DM)_q$). They are such that :

(i) $(\mathrm{id} \underline{\otimes} \varepsilon') \circ (\nu' \underline{\otimes} \mathrm{id}) = \mathrm{id}$;

(ii) $(\varepsilon' \underline{\otimes} \mathrm{id}) \circ (\mathrm{id} \underline{\otimes} \nu') = \mathrm{id}$.

So the proof is finished.

*Remark.* — $\mathrm{Hom}_C({}_BM, {}_BN) = \mathrm{Hom}_S({}_BM, {}_BN)$.

We have then the following result :

THEOREM 5.5. — *If $M.$ and $N.$ are mixed complexes and $N.$ is such that $DDN.$ is homotopically equivalent to $N.$, there is an isomorphism :*

$$HC^n(M., N.) \cong HC^n(M. \otimes DN.).$$

*Proof.* — By the previous Proposition, $DN.$ is also a mixed complex and ${}_BDN$, being the strong dual of ${}_BN$ verifies :

$$\begin{aligned} HC^n(M., N.) &= H_{-n}(\mathrm{Hom}_S(k[u] \underline{\otimes} M., k[u] \underline{\otimes} N.) \\ &= H_{-n}(\mathrm{Hom}_S((k[u] \underline{\otimes} M.) \,\square_{k[u]}\, (k[u] \underline{\otimes} DN.), k[u]) \\ &= H_{-n}(\mathrm{Hom}_S(k[u] \underline{\otimes} (M. \otimes DN.), k[u]), \end{aligned}$$

and this, by definition, is $HC^n(M. \otimes DN.)$.

The last isomorphism is due to Eilenberg-Moore [E-M].

COROLLARY 5.6. — *If $M.$ has the homotopy type of a projective finitely generated chain complex, there is an isomorphism :*

$$HC^n(M., N.) \approx HC^-_{-n}(DM. \otimes N.)$$

COROLLARY 5.7. — *If $X$ and $Y$ are pointed paces provided of a pointed action of $S^1$, and $Y$ is homotopically equivalent to $DDY$, then $\overline{HC}^n(\mathbb{S}.X, \mathbb{S}.Y)$ is isomorphic to $\overline{H}^n_{S^1}(X \wedge DY)$.*

*Proof.* — In this conditions, $\mathcal{S}.(DY)$ is the dual of $\mathcal{S}.(Y)$ in $\mathrm{Ho}(\partial\text{-mod}_k)$ and so :

$$\begin{aligned}\overline{HC}^n(\mathbb{S}.X, \mathbb{S}.Y) &\cong HC^n(\mathcal{S}.X, \mathcal{S}.Y) \cong HC^n(\mathcal{S}.X \otimes \mathcal{S}.DY)\\ &\cong HC^n\big(\mathcal{S}.(X \wedge DY)\big) \cong \overline{H}^n_{S^1}(X \wedge DY).\end{aligned}$$

## 6. Bivariant $S^1$-equivariant cohomology and bivariant cyclic cohomology

In [C], CRABB defines the bivariant $S^1$-equivariant cohomology of two $CW$-complexes equipped with an $S^1$-action, $X$ and $Y$, as :

$$H^i_{S^1}(X, Y) = H^i_B\big(ES^1 \times_{S^1} X, ES^1 \times_{S^1} Y\big)$$

and this last object is defined as

$$\varinjlim_k \Big[(B \times S^k) \wedge_B (ES^1 \times_{S^1} X), (ES^1 \times_{S^1} Y) \wedge_B (B \times \mathbb{H})_{k+i}\Big]_B$$

(the homotopy classes of morphisms that commute with the projections) where $B = BS^1$ and given two fibrations $Z \to BS^1$ and $Z' \to BS^1$ with sections $s$ and $s'$, $Z \wedge_B Z'$ is the push-out of the diagram :

$$\begin{array}{ccc} BS^1 & \xrightarrow{\ s\ } & Z \\ {\scriptstyle s'}\big\downarrow & & \big\downarrow \\ Z' & \longrightarrow & Z \wedge_B Z'. \end{array}$$

From now on, if $X$ is an $S^1$ space, we denote $\mathbb{X} = ES^1 \times_{S^1} X$.

If $D_{BS^1}(Y)$ is the $B$-dual [D-P, 6] of $\mathbb{Y}$, then the last expression in the definition of $H^i_{S^1}(X, Y)$ is :

$$\varinjlim_k \big[S^k \wedge (\mathbb{X} \wedge_B D_B(\mathbb{Y})/B), \mathbb{H}_{k+i}\big] = \overline{H}^i\big((\mathbb{X} \wedge_B D_B(\mathbb{Y})/B\big).$$

But, if $DY$ is the dual of $(Y, 0)$ in Stab, following ([B-G, 4]), then $ES^1 \times_{S^1} DY$ is canonically isomorphic to the $B$-dual of $\mathbb{Y}$, that will be denoted $\mathbb{DY}$, so :

$$H^i_{S^1}(X, Y) = \overline{H}^i\big((\mathbb{X} \wedge_B \mathbb{DY})/B\big).$$

This is a reduced cohomology, in the sense that if $X = Y = \mathrm{pt}$, then $H^i_{S^1}(X, Y) = 0$, for every $i$.





PROPOSITION 6.1. — $\overline{H}^n(\mathbb{X} \wedge_B \mathbb{DY}) = \overline{H}^n_{S^1}(X \wedge DY)$ *and we have exact sequences (for $k \in \mathbb{Z}$)*

$$\begin{aligned} 0 \to H^{2k}\big((\mathbb{X} \wedge_B \mathbb{DY})/B\big) &\longrightarrow H^{2k}(\mathbb{X} \wedge_B \mathbb{DY}) \xrightarrow{\ i'\ } H^{2k}(B)\\ &\longrightarrow H^{2k+1}\big((\mathbb{X} \wedge_B \mathbb{DY})/B\big) \longrightarrow H^{2k+1}(\mathbb{X} \wedge_B \mathbb{DY}) \to 0,\end{aligned}$$

*where $i'$ splits by means of the section $s : B \to \mathbb{X} \wedge_B \mathbb{DY}$.*

*Proof.* — We observe that $\mathbb{X} \wedge_B \mathbb{DY} \to B$ is a fibration of fibre $X \wedge DY$, so we have the long Gysin exact sequence for this fibration.

There is another fibration $ES^1 \times_{S^1} (X \wedge DY) \to B$ also of fibre $X \wedge DY$, and a morphism $F : ES^1 \times_{S^1} (X \wedge DY) \to \mathbb{X} \wedge \mathbb{DY}$ given by $F\big((e, x, y')\big) = \big((e, x), (e, y')\big)$ which induces the identity on the fibre (where $(e, x)$ is the class of (e, x)). Then $H^n(\mathbb{X} \wedge_B \mathbb{DY}) = H^n_{S^1}(X \wedge DY)$ for $n \in \mathbb{N}$ and we also have the corresponding isomorphisms for the reduced cohomology theory.

Next, if we regard (using the section $s : B \to \mathbb{X} \wedge_B \mathbb{DY}$), $B$ as a subspace of $\mathbb{X} \wedge_B \mathbb{DY}$, whose quotient space is $(\mathbb{X} \wedge_B \mathbb{DY})/B$ we get the desired exact sequences.

From now on we shall take $k = \mathbb{Z}$.

THEOREM 6.2. — *Under the conditions of Theorem 5.5, there is an isomorphism :*

$$\overline{HC}^n(\mathbb{S}.X, \mathbb{S}.Y) \cong H^n_{S^1}(X, Y).$$

*Proof.* — By COROLLARY 6.7, $\overline{HC}^n(\mathbb{S}.X, \mathbb{S}.Y) = HC^n(\mathcal{S}.X, \mathcal{S}.Y)$ is isomorphic to $H^n_{S^1}(X \wedge DY)$, and by the first part of PROPOSITION 6.1, the latter is $\overline{H}^n(\mathbb{X} \wedge_B \mathbb{DY})$.

Then, using the exact sequences of this proposition, we obtain that :

$$H^n(\mathbb{X} \wedge_B \mathbb{DY}) = \begin{cases} H^n\big((\mathbb{X} \wedge_B \mathbb{DY})/B\big) & \text{if } n = 2k+1,\\ H^n\big((\mathbb{X} \wedge_B \mathbb{DY})/B\big) \oplus \mathbb{Z} & \text{if } n = 2k. \end{cases}$$

Therefore, $\overline{H}^n(\mathbb{X} \wedge_B \mathbb{DY})$ is isomorphic to $H^n\big((\mathbb{X} \wedge_B \mathbb{DY})/B\big)$, which is, by definition, $H^n_{S^1}(X, Y)$.

*Example.* — If the action of $S^1$ on $Y$ is trivial, we observe that $\mathbb{Y} = BS^1 \times Y$ and $H^n(\mathbb{X}, Y)$ is $H^n_B(\mathbb{X}, \mathbb{Y})$ because :

$$H^n(\mathbb{X}, Y) = \varinjlim_k \big[\Sigma^k \mathbb{X}, Y \wedge \mathbb{H}_{k+n}\big],$$

while

$$\begin{aligned} H^n_B(\mathbb{X},\mathbb{Y}) &= \varinjlim_k \left[\Sigma^k_B \mathbb{X}, (B\times Y)\wedge_B (B\times \mathbb{H})_{k+n}\right]_B \\ &= \varinjlim_k \left[\Sigma^k_B \mathbb{X}, B\times (Y\wedge \mathbb{H})_{k+n}\right]_B \\ &= \varinjlim_k \left[\Sigma^k \mathbb{X}, Y\wedge \mathbb{H}_{k+n}\right] \qquad \text{(see [C-S])}. \end{aligned}$$

If we consider now the structure of $\mathbb{S}.Y$ as a cyclic module, we find that it is trivial, and it is known then that $HC_*(\mathbb{S}.Y) = k[u]\otimes H_*(\mathbb{S}.Y)$, with trivial $S$ coaction on $H_*(\mathbb{S}.Y)$.

In this case, $HC^n(\mathbb{S}.X,\mathbb{S}.Y) = \operatorname{Hom}\big(HC_*(\mathbb{S}.X), H_*(\mathbb{S}.Y)\big)_{-n}$ and this is $\operatorname{Hom}\big(H_*(X), H_*(\mathbb{S}.Y)\big)_{-n}$ (cf. [J-K, § 7]).

Taking the reduced bivariant cyclic cohomology, we get the following identification :

$$\overline{H}C^n(\mathbb{S}.X,\mathbb{S}.Y) = \operatorname{Hom}\big(\overline{H}_*(X), \overline{H}_*(\mathbb{S}.Y)\big)_{-n}.$$

By the isomorphism of the last theorem, $\overline{H}C^n(\mathbb{S}.X,\mathbb{S}.Y)$ is $H^n(\mathbb{X},Y)$, which is also $\operatorname{Hom}\big(\overline{H}_*(\mathbb{X}), \overline{H}_*(Y)\big)_{-n}$ by using the short exact sequence of (1) if we suppose, as in [J-K, § 4], that $HC_*(\mathbb{S}.X) = H_*(\mathbb{X})$ is $k$-projective.

## 7. Connes' long exact sequence

Kassel has shown [K2, I.2.3] that there is a long exact sequence (l.e.s.) for bivariant cyclic cohomology :

If $M.$ and $N.$ are cyclic $k$-modules and $M$ is $k$-projective, there is a long exact sequence :

$$\cdots \to HC^{n-2}(M.,N.) \xrightarrow{S} HC^n(M.,N.) \xrightarrow{I} HH^n(M.,N.) \xrightarrow{B} HC^{n-1}(M.,N.) \to \cdots$$

and he has also described the morphisms $S, B$ and $I$.

There is also a long (Gysin) exact sequence :

$$\cdots \to H^{n-2}_{S^1}(X\wedge DY) \xrightarrow{S'} H^n_{S^1}(X\wedge DY) \xrightarrow{I'} H^n(X\wedge DY) \xrightarrow{B'} H^{n-1}_{S^1}(X\wedge DY) \to \cdots$$

and we want to show that, taking $M. = \mathbb{S}.X$ and $N. = \mathbb{S}.Y$, if we relate the reduced versions of both sequences by the isomorphism of the above paragraphs, then the diagram is commutative :





Proposition 7.1. — *If $X$ and $Y$ are $S^1$-spaces satisfying the conditions of Theorem 5.5, then the following diagram is commutative :*

$$\begin{tikzcd} \cdots \arrow[r] & \overline{H}C^{n-2}(\mathbb{S}.X,\mathbb{S}.Y) \arrow[r, "S"] \arrow[d, "\phi_{n-2}"] & \overline{H}C^{n}(\mathbb{S}.X,\mathbb{S}.Y) \arrow[d, "\phi_{n}"] \\ \cdots \arrow[r] & H^{n-2}_{S^1}(X,Y) \arrow[r] & H^{n}_{S^1}(X,Y) \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r, "I"] & \overline{H}\overline{H}^{n}(\mathbb{S}.X,\mathbb{S}.Y) \arrow[r, "B"] \arrow[d, "\phi'n"] & \overline{H}C^{n-1}(\mathbb{S}.X,\mathbb{S}.Y) \arrow[r] \arrow[d, "\phi_{n-1}"] & \cdots \\ {} \arrow[r] & H^{n}(X,Y) \arrow[r] & H^{n-1}_{S^1}(X,Y) \arrow[r] & \cdots. \end{tikzcd}$$

*Proof.* — We can introduce an additional row in the middle and consider the following diagram :

$$\begin{tikzcd} \cdots \arrow[r] & \overline{H}\overline{H}^{n}(\mathbb{S}.X,\mathbb{S}.Y) \arrow[r] \arrow[d] & \overline{H}C^{n-1}(\mathbb{S}.X,\mathbb{S}.Y) \arrow[d] \\ \cdots \arrow[r] & \overline{H}^{n}(\mathbb{S}.(X\wedge DY)) \arrow[r] \arrow[d] & \overline{H}C^{n-1}(\mathbb{S}.(X\wedge DY)) \arrow[d] \\ \cdots \arrow[r] & H^{n}(X,Y) \arrow[r] & H^{n-1}_{S^1}(X,Y) \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r] & \overline{H}C^{n+1}(\mathbb{S}.X,\mathbb{S}.Y) \arrow[r] \arrow[d] & \overline{H}\overline{H}^{n+1}(\mathbb{S}.X,\mathbb{S}.Y) \arrow[r] \arrow[d] & \cdots \\ {} \arrow[r] & \overline{H}C^{n+1}(\mathbb{S}.(X\wedge DY)) \arrow[r] \arrow[d] & \overline{H}^{n+1}(\mathbb{S}.(X\wedge DY)) \arrow[r] \arrow[d] & \cdots \\ {} \arrow[r] & H^{n+1}_{S^1}(X,Y) \arrow[r] & H^{n+1}(X,Y) \arrow[r] & \cdots. \end{tikzcd}$$

As the lower part commutes [J, thm 3.3]), we have to show that the upper part also commutes.

The first exact sequence is a consequence of [K2, I, prop. 2.1], while the second one follows from the short exact sequence :

$$0 \to \mathrm{Ker}(\underline{\mathrm{AdS}})_{-n+2} \to \mathrm{Ker}(\underline{\mathrm{AdS}})_{-n} \to \mathrm{Hom}_{-n}\big(\mathbb{S}.(X \wedge DY), k\big) \to 0,$$

where

$$\underline{\mathrm{AdS}} : \mathrm{Hom}_{-n+2}\big(k[u] \underline{\otimes} \mathbb{S}.(X \wedge DY), k[u]\big) \longrightarrow \mathrm{Hom}_{-n}(k[u] \underline{\otimes} \mathbb{S}.(X \wedge DY), k[u])$$

is defined by $(\underline{\mathrm{AdS}})(f) = S \circ f - f \circ S$.

So, the proof reduces to verify the commutativity of the following square, as the maps $\phi_i$, $\phi_i'$ are defined between the complexes before taking homology

$$\begin{tikzcd}
\mathrm{Ker}(\mathrm{AdS})_{-n+2} \arrow[r] \arrow[d, "\phi'_{n-2}"'] & \mathrm{Ker}(\mathrm{AdS})_{-n} \arrow[d, "\phi'_n"] \\
\mathrm{Ker}(\underline{\mathrm{AdS}})_{-n+2} \arrow[r] & \mathrm{Ker}(\underline{\mathrm{AdS}})_{-n}.
\end{tikzcd}$$

We observe that if $f$ is an element of $\mathrm{Hom}_{-n+2}\big(k[u] \underline{\otimes} \mathcal{S}.X, k[u] \underline{\otimes} \mathcal{S}.Y\big)$ then $\phi'_{n-2}(f) = \big(\mathrm{id} \underline{\otimes} \varepsilon\big) \circ \big(f \,\square_{k[u]}\, \mathrm{id}_{\mathcal{S}.(DY)}\big)$.

So, if $f : \big(k[u] \underline{\otimes} \mathcal{S}.X\big)_j \to \big(k[u] \underline{\otimes} \mathcal{S}.Y\big)_{j-n+2}$ and

$$f \in \mathrm{Ker}(\mathrm{AdS}), \quad \phi'_{n-2}(f) \varepsilon \, \mathrm{Ker}(\mathrm{AdS}),$$

then we have that :

$$\begin{tikzcd}
f \arrow[r] \arrow[d] & (Sf) \arrow[d] \\
\big(\mathrm{id} \underline{\otimes} \varepsilon\big) \circ \big(f \,\square_{k[u]}\, \mathrm{id}\big) \arrow[r] & S\phi'_{n-2}(f) = \phi'_n(Sf)
\end{tikzcd}$$

is commutative.





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