α-DERIVATIONS

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Abstract. Let A be a commutative k-algebra with 1. We present a characterization of α-derivations, for $\alpha\colon A \to A$ a morphism of algebras, using α-Taylor series. When $S = \mathbf{C}[x, x^{-1}, \xi]$ and $\alpha(x) = qx$, $\alpha(\xi) = q\xi$, we compare the q-de Rham cohomology of the $\mathbf{C}$-algebra S with the Hochschild homology of D_q, the algebra of q-difference operators on $\mathbf{C}[x, x^{-1}]$, for $q \in \mathbf{C}$, $\quad q \neq 0, 1$.

α-Dérivations

Résumé. Soient k et A deux anneaux commutatifs unitaires, A une k-algèbre. Etant donné un endomorphisme α de l'algèbre A, nous montrons une caracterisation des α-dérivations en utilisant les α-séries de Taylor, dont nous prouvons certaines propriétés. Dans le cas particulier de l'algèbre D_q des operateurs q-differentiels sur $\mathbf{C}[x, x^{-1}]$ nous faisons la comparaison entre la q-cohomologie de De Rham de $\mathbf{C}[x, x^{-1}, \xi]$, et de homologie d'Hochschild de D_q, $q \in \mathbf{C}$, $\quad q \neq 0, 1$.

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Version francaise abregée. Soient k et A deux anneaux commutatifs unitaries, a une k-algèbre et α un endomorphisme de A. Compte tenue de la définition d'une α-dérivation de A a valeurs dans un A-bimodule M [5], nous caracterisons le A-module des α-dérivations $D_k^\alpha(A, M)$ à l'aide des applications de Taylor "tordues", $T_\alpha\colon A \to A \otimes A$ ou $T_\alpha(a) = 1 \otimes a - \alpha(a) \otimes 1$.

Nous etablissons aussi un rapport entre les α-dérivations et l'homologie d'Hochschild tordue [4] $HH^\alpha(A, M)$. Ensuite, nous prouvons que l'application T_α est universelle parmi les k-α-séries de Taylor définies avant, et nous étudions le comportement de l'A-module des α-différentiels par rapport a la localisation. Finalement nous faisons une comparaison entre l'homologie d'Hochschild des opérateurs q-différentiels sur $A = \mathbf{C}[x, x^{-1}]$, $q \in \mathbf{C}$, $\quad q \neq 0, 1$, et la q-cohomologie de De Rham de $\mathbf{C}[x, x^{-1}, \xi]$ [5].

1. Introduction

Let A be a commutative k-algebra with 1, $\alpha\colon A \to A$ a morphism of algebras. We recall the definition of α-derivations of A into A-bimodules M, and present a characterization of them, using the analogies with the theory of derivations. We also relate them to "twisted" Hochschild homology.

In **Section 2** we recall what a derivation is and we define α-derivations and α-Taylor operators, which allows us to characterize the module of α-differentials $\Omega_k^\alpha(A)$. In **Section 3** we define k-α-Taylor series, and show that the α-Taylor operator is universal for k-α-Taylor series. We introduce the algebra D_q of q-difference operators on $\mathbf{C}[x, x^{-1}]$ in **Section 4**, and compare the q-de Rham cohomology of the $\mathbf{C}$-algebra $S = \mathbf{C}[x, x^{-1}, \xi]$ with the Hochschild homology of D_q.

2. α-Derivations and the α-Taylor operator

Let A be an associative k-algebra with 1 and $\alpha\colon A \to A$ a morphism of algebras. We shall present in the following section a characterization of α-derivations, using the analogies with the theory developed for derivations from A into an A-bimodule M. We recall that d is a derivation of A into an A-bimodule M if and only if d is a k-linear map, $d\colon A \to M$, such that

$$d(xy) = xd(y) + d(x)y$$

and $D_k(A, M)$ denotes the set of all derivations $d\colon A \to M$ (in fact, it is a k-module, and if A is commutative, it is an A-bimodule). If A is commutative and $A \otimes A^{op}$ is considered an A-bimodule by $a(b \otimes c)d = adb \otimes c$, we have the Taylor operator [7] $T\colon A \to A \otimes A^{op}$ defined by $T(a) = 1 \otimes a - a \otimes 1$, and the multiplication map, $\mu\colon A \otimes A^{op} \to A$, given by $\mu(a \otimes b) = ab$. One easily verified consequence of these definitions is that

$$0 \longrightarrow I \longrightarrow A \otimes A^{op} \longrightarrow A \longrightarrow 0$$

is a short exact sequence of A-modules, where I is the ideal in $A \otimes A^{op}$ generated by $\{T(a), a \in A\}$. Next we observe that the Taylor operator is not a derivation, but

$$T(ab) = aT(b) + T(a)b + T(a)T(b).$$

Therefore, it seems reasonable to consider I/I^2 as the module of 1-differentials $\Omega^1_{A/k}$. Summarizing this discussion, we recall the following important result form [6] :

Proposition 2.1. *For an A-module M, there is a canonical A-isomorphism*

$$Hom_A(I/I^2, M) \longrightarrow Der_k(A, M)$$

In particular, the isomorphism

$$Hom_A(I/I^2, A) \cong Der_k(A)$$

identifies the derivation module of A canonically with the dual of the differential module of A.

When A is commutative, the Hochschild homology of A, $HH_1(A)$ is isomorphic, as an A-module to $\Omega^1_{A/k}$. If A is not commutative, $HH_1(A)$ is considered instead of $\Omega^1_{A/k}$, which is not defined in this case.

We are now in a position to make the following definitions.

Definition 2.2. Assume A is commutative, $\alpha\colon A \to A$ is a morphism of algebras and M is an A-bimodule that verifies $ma = \alpha(a)m$, for $m \in M, a \in A$.

a) d_α is an α-derivation of A into M if d_α is a k-linear map, $d_\alpha\colon A \to M$, such that
$$d_\alpha(ab) = \alpha(a)d_\alpha(b) + d_\alpha(a)b \qquad \text{for } a, b \in A.$$

b) $D^\alpha_k(A, M)$ denotes the set of all α-derivations $d_\alpha\colon A \to M$.

c) We shall denote by $\Omega_k^\alpha(A)$ the module of α-differentials in A, ie, the A-module satisfaying:
   i) there is an α-derivation $D_\alpha\colon A \to \Omega_k^\alpha(A)$,
   ii) $\Omega_k^\alpha(A)$ is generated by $\{D_\alpha(a)\ , a \in A\}$ over A,
   iii) for any α-derivation $d_\alpha\colon A \to M$, there exists a unique A-linear map $h\colon \Omega_k^\alpha(A) \to M$ such that $d_\alpha = h \circ D_\alpha$.

Remark 2.3. $D_k^\alpha(A, M)$ is an A-bimodule that verifies $da = \alpha(a)d$, for $a \in A, d \in D_k^\alpha(A, M)$.

Now we define the α-Taylor operator $T_\alpha\colon A \to A \otimes A^{op}$ given by

$$T_\alpha(P) = 1 \otimes P - \alpha(P) \otimes 1$$

and we denote by $\mu_\alpha\colon A \otimes A^{op} \to A$ the α-multiplication map $\mu_\alpha(P \otimes Q) = P\alpha(Q)$. As an immediate consequence of these definitions, we have the following easily checked proposition.

Proposition 2.4.

$$0 \longrightarrow I_\alpha \longrightarrow A \otimes A^{op} \xrightarrow{\mu_\alpha} A \longrightarrow 0$$

is a short exact sequence of A-modules, where I_α is the ideal in $A \otimes A^{op}$ generated by $\{T_\alpha(a)\}$ where a ranges over A.

The following facts concerning the α-Taylor operator are easily verified, when $A \otimes A^{op}$ is considered an A-bimodule by

$$a(b \otimes c)d = \mu_\alpha(ad)b \otimes c = a\alpha(d)b \otimes c.$$

Properties 2.5.

a) T_α *is k-linear*
b) $T_\alpha(PQ) = (\alpha(P) \otimes 1)T_\alpha(Q) + T_\alpha(P)(1 \otimes Q) =$
   $\alpha(P)T_\alpha(Q) + T_\alpha(P)Q + T_\alpha(P)T_\alpha(Q)$
c) $T_\alpha(P^n) = \sum_{i=0}^{n-1} \left(\alpha(P^{n-i-1}) \otimes P^i\right) T_\alpha(P)$
d) $T_\alpha(P_1 \dots P_n) =$
   $\sum_{k=1}^{n-1} (-1)^{k+1} \left(\sum_{i_1 < \cdots < i_k} \alpha(P_{i_1} \dots P_{i_k}) T_\alpha(P_1 \dots \widehat{P}_{i_j} \dots P_n)\right) +$
   $T_\alpha(P_1)T_\alpha(P_2) \dots T_\alpha(P_{n-1})T_\alpha(P_n)$

Remark 2.6. $A \otimes A^{op}$ has two module structures, given by the ring homomorphisms $A \to A \otimes A^{op}$, $a \to \alpha(a) \otimes 1$, and $A \to A \otimes A^{op}$, $a \to 1 \otimes a$. They induce two A-module structures onI_α. From those, we get induced A-module structures of I_α/I_α^2 which, however, coincide in I_α/I_α^2 since

$$(1 \otimes r - \alpha(r) \otimes 1)(1 \otimes a - \alpha(a) \otimes 1 + I_\alpha^2) \in I_\alpha^2.$$

In view of **Properties 2.5**, we can reformulate **Proposition 2.1** for α-derivations.

Proposition 2.7.

a) $\Omega_k^\alpha(A) \cong I_\alpha/I_\alpha^2$

b) *For an A-module M, there is a canonical A-isomorphism of A-bimodules*

$$Hom_A(I_\alpha/I_\alpha^2, M) \longrightarrow D_k^\alpha(A, M)$$

Proof. To prove the first statement, we shall verify i), ii) and iii) of **Definition 2.2 c)**.

i) The mapping $D_\alpha\colon A \to I_\alpha/I_\alpha^2$, defined by the composition

$$A \xrightarrow{T_\alpha} I_\alpha \xrightarrow{\pi} I_\alpha/I_\alpha^2$$

is an α-derivation (using **Property 2.5 b)**).

ii) It's obvious that I_α/I_α^2 is generated by $\{T_\alpha(a) + I_\alpha^2, \quad a \in A\}$ over A.

iii) Let $d_\alpha\colon A \to M$ be an α-derivation. We define $\Theta\colon A \otimes A^{op} \to M$ by $\Theta(a \otimes b) = ad_\alpha(b)$. Now, a direct computation shows that $\Theta(I_\alpha^2) = 0$. So there exists a unique A-linear map $h\colon I_\alpha/I_\alpha^2 \to M$ such that $h \circ \pi = \Theta/I_\alpha$.
Finally, $h \circ D_\alpha(a) = \Theta(T_\alpha(a)) = d_\alpha(a)$. Since the A-module I_α/I_α^2 is generated by $\{D_\alpha(a), \quad a \in A\}$, there can be only one mapping h such that $h \circ D_\alpha = d_\alpha$. Hence D_α is universal.

The second statement of the Proposition is only a reformulation of the universal property of $D_\alpha\colon A \to I_\alpha/I_\alpha^2$. □

Remark 2.8. Given α, one may also consider "twisted Hochschild homology" and "twisted Cyclic homology", which differs from ordinary Hochschild homology by the face and cyclic operators, which now involves the action of the automorphism. The twisted theory appeared implicitely in [8] and [9], and explicitely in [4].

Explicitely, if $\alpha\colon A \to A$ is an automorphism of algebras, we have:

$d_i\colon A^{\otimes(n+1)} \to A^{\otimes n}, \quad for \quad 0 \le i \le n$

$s_i\colon A^{\otimes(n+1)} \to A^{\otimes(n+2)}, \quad for \quad 0 \le i \le n$

$t\colon A^{\otimes(n+1)} \to A^{\otimes(n+1)}$

defined by

$$d_i(a_0 \otimes \ldots \otimes a_n) = \begin{cases} (a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n) & if \quad i < n, \\ (\alpha(a_n) a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}) & if \quad i = n. \end{cases}$$

$$s_i(a_0 \otimes \ldots \otimes a_n) = (a_0 \otimes \ldots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \ldots \otimes a_n)$$

$$t(a_0 \otimes \ldots \otimes a_n) = (\alpha(a_n) \otimes a_0 \otimes \ldots \otimes a_{n-1})$$

which verifies the relations:

$d_i d_j = d_{j-1} d_i, \quad for \quad i < j$

$$d_i t = \begin{cases} t d_{i-1} & for \quad 1 \le i \le n, \\ d_n, & for \quad i = 0. \end{cases}$$

$s_i s_j = s_{j+1} s_i \quad for \quad i \le j$

$$d_i s_j = \begin{cases} s_{j-1} d_i & for \quad i < j, \\ id & for \quad i = j, j+1, \\ s_j d_{i-1} & for \quad i > j+1. \end{cases}$$

$$s_i t = \begin{cases} t s_{i-1} & for \quad 1 \leq i \leq n, \\ t^2 s_n & for \quad i = 0. \end{cases}$$

$$t_n^{n+1}(a_1 \otimes \ldots \otimes a_n) = (\alpha(a_1) \otimes \ldots \otimes \alpha(a_n))$$

Writing $b^\alpha = \sum_{i=0}^n (-1)^i d_i$, we verify that $HH_1^\alpha(A) = \frac{Ker(A \otimes A \xrightarrow{b^\alpha} A)}{Im(A \otimes A \otimes A \xrightarrow{b^\alpha} A \otimes A)}$ is isomorphic to I_α / II_α, by the map

$$I_\alpha / II_\alpha \to HH_1^\alpha(A)$$

$$\overline{T_\alpha(a)} \to [1 \otimes a]$$

So we can view the difference between ordinary Hochschild homology and twisted Hochschild homology in terms of the difference between I and I_α.

Now, we consider $\alpha - n$-derivations, ie, k-linear maps $d_\alpha^n \colon A \to M$ such that $d_\alpha^n(x_0 \ldots x_n) = \sum_{k=1}^n (-1)^{k+1} \left(\sum_{i_1 < \cdots < i_k} \alpha(x_{i_1} \ldots x_{i_k}) d_\alpha^n(x_0 \ldots \widehat{x}_{i_j} \ldots x_n) \right)$

and we denote by $\Omega_\alpha^{k,n}(A)$ the module of α-n-differentials in A.

One easily verified consequence of **Property 2.5** is the following proposition.

Proposition 2.9. *$\Omega_{A/k}^{\alpha,n}$ is isomorphic to I_α / I_α^n.*

3. α-Taylor series.

Definition 3.1. Suppose C is an A-algebra, B is an A-subalgebra of C and $\alpha \colon A \to A$ is a morphism of algebras.

a) We say that $\eta_\alpha \colon A \to B$ is an B-valued k-α-Taylor serie if:
   i) η_α is k-linear and $\eta_\alpha(1) = 0$
   ii) $\eta_\alpha(ab) = \alpha(a)\eta_\alpha(b) + \alpha(b)\eta_\alpha(a) + \eta_\alpha(a)\eta_\alpha(b)$ for a,b $\in A$

b) Suppose B, C are k-algebras, $B \subseteq C$, $\alpha \colon C \to C$ a morphism of algebras. We define $I_\alpha(C/B)$ by the exact sequence

$$0 \longrightarrow I_\alpha(C/B) \longrightarrow C \otimes_B C^{op} \xrightarrow{\mu_\alpha} C \longrightarrow 0.$$

We now prove some results concerning α-Taylor series.

Proposition 3.2. *The map $u\colon A \to I_\alpha(A/k)$ given by $a \to T_\alpha(a)$, is universal for k-α-Taylor series (T_α is the k-α-Taylor operator defined in* **Section 2***).*

Proof. It's obvious that the map $u\colon A \to I_\alpha(A/k)$ is an $I_\alpha(A/k)$-valued k-α-Taylor serie. Let $\eta_\alpha\colon A \to B$ be a B-valued k-α-Taylor serie. We only have to show that there exists a unique A-morphism of algebras $\psi\colon I_\alpha(A/k) \to B$ such that $\psi \circ u = \eta_\alpha$. The given map $\eta_\alpha\colon A \to B$ is k-linear, therefore we may define an A-linear map $h\colon A \otimes A^{op} \to B$ by $h(a \otimes b) = a\eta_\alpha(b)$. It follows that $h(T_\alpha(a)) = \eta_\alpha(a)$ since $\eta_\alpha(1) = 0$. Clearly, the restriction of h to $I_\alpha(A/k)$ satisfies the conditions requiered. Notice that uniqueness is immediate from the fact that $\{T_\alpha(a), a \in A\}$ generates $I_\alpha(A/k)$ as an A-module. To complete the proof, we only have to show that h is a morphism of algebras. To this end we observe that is sufficies to verify that

$$h\left(\prod_{j=1}^{s}(1 \otimes x_j - \alpha(x_j) \otimes 1)\right) = \prod_{j=1}^{s} h(1 \otimes x_j - \alpha(x_j) \otimes 1)$$

because h is A-linear. This may be easily done by induction. In case $s = 1$, there is nothing to prove. Assume the equality for $r < s$. We first observe that

$\prod_{j=1}^{s-1}(1 \otimes x_j - \alpha(x_j) \otimes 1) = \sum u_i \otimes v_i \in I_\alpha(A/k)$

so $\sum u_i\alpha(v_i) = 0$. On the other hand,

$h\left(\prod_{j=1}^{s-1}(1 \otimes x_j - \alpha(x_j) \otimes 1)(1 \otimes x_s - \alpha(x_s) \otimes 1)\right) =$

$h\left((\sum_i u_i \otimes v_i)(1 \otimes x_s - \alpha(x_s) \otimes 1)\right) = \sum_i \left(u_i\ \eta_\alpha(x_s v_i) - u_i\ \alpha(x_s)\ \eta_\alpha(v_i)\right) =$

$\sum_i \left(u_i \left(\alpha(x_s)\ \eta_\alpha(v_i) + \alpha(v_i)\ \eta_\alpha(x_s) + \eta(v_i)\ \eta_\alpha(x_s)\right) - u_i\ \alpha(x_s)\ \eta_\alpha(v_i)\right) =$

$\sum_i \left(u_i\ \alpha(v_i) + u_i\ \eta_\alpha(v_i)\right) \eta_\alpha(x_s) = \sum_i u_i\ \eta_\alpha(v_i)\ \eta_\alpha(x_s) =$

$h\left(\sum_i u_i \otimes v_i\right)\ h(1 \otimes x_s - \alpha(x_s) \otimes 1)$. ☐

Proposition 3.3. *Suppose A, B are k-algebras such that $A \subseteq B$, and $\alpha\colon B \to B$ is a morphism of algebras such that $\alpha/A = id$. There exists an exact sequence of B-algebras*

$$0 \longrightarrow N_\alpha(A/k) \longrightarrow I_\alpha(B/k) \xrightarrow{\Theta} I_\alpha(B/A) \longrightarrow 0$$

where $N_\alpha(A/k)$ is generated as an ideal in $I_\alpha(B/k)$ by the elements $(1 \otimes x - \alpha(x) \otimes 1), x \in A$, and Θ is the restriction of the map $B \otimes_k B^{op} \to B \otimes_A B^{op}$ to the ideal $I_\alpha(B/k)$.

Proof. The map Θ is clearly onto. We shall prove that $I_\alpha(B/k)/N_\alpha(A/k)$ is universal for A-α-Taylor series on B, from which the assertion of the theorem follows immediately. First note that the map $\eta_\alpha\colon B \to I_\alpha(B/k)/N_\alpha(A/k)$ defined as the composition

$$\eta_\alpha\colon B \xrightarrow{T_\alpha} I_\alpha(B/k) \xrightarrow{proj} I_\alpha(B/k)/N_\alpha(A/k)$$

satisfy the conditions:

$$\eta_\alpha(xy) = \alpha(x)\eta_\alpha(y) + \alpha(y)\eta_\alpha(x) + \eta_\alpha(x)\eta_\alpha(y), \quad \text{for } x, y \in B.$$

If $a \in A$, then $\eta_\alpha(ay) = \alpha(a)\eta_\alpha(y) + \alpha(y)\eta_\alpha(a) + \eta_\alpha(a)\eta_\alpha(y) = a\eta_\alpha(y)$ since $\eta_\alpha(a) \in N_\alpha(A/k)$ and $\alpha/A = id$. As $\eta_\alpha(1) = 0$, it follows that η_α is an $I_\alpha(B/k)/N_\alpha(A/k)$-valued A-α-Taylor serie on B. Now, suppose that $\rho_\alpha\colon B \to R$ is an R-valued A-α-Taylor serie. For $x \in k$, $\quad x1 \in A$ (because A is a k-algebra). Therefore ρ_α is an R-valued k-α-Taylor serie, and thus there exists a unique B-algebra morphism $h\colon I_\alpha(B/k) \to R$ such that $\rho_\alpha = h \circ T_\alpha$. Since $\rho_\alpha(a1) = a\rho_\alpha(1) = 0$ for each $a \in A$, $(h \circ T_\alpha)/A = 0$. Therefore the kernel of h contains $N_\alpha(A/k)$, and h factors uniquely through $I_\alpha(B/k)/N_\alpha(A/k)$. $\square$

Remark 3.4. The module $I_\alpha(A/k)$ has one outstanding drawback, which is that if S is a multiplicatively closed subset of A, then in general, $A_S \otimes_A I_\alpha(A/k) \not\cong I_\alpha(A_S/k)$, i.e., I_α doesn't localize. For example, $\alpha = id$, $A = k[x]$ and $S = \{1, x, x^2, \dots\}$.

Theorem 3.5. *Let S be a multiplicatively closed subset of A, $1 \in S$, $0 \notin S$. Then*

$$I_\alpha(A_S/k)/I_\alpha(A_S/k)^2 \cong A_S \otimes_A I_\alpha(A/k)/I_\alpha(A/k)^2$$

Proof. The mapping $d_\alpha\colon A_S \to A_S \otimes_A I_\alpha(A/k)/I_\alpha(A/k)^2$ defined by

$d_\alpha(\frac{a}{s}) = -\frac{\alpha(a)}{\alpha(s^2)} \otimes \overline{T_\alpha(s)} + \frac{1}{\alpha(s)} \otimes \overline{T_\alpha(a)}$

is an α-derivation:

$d_\alpha(\frac{a_1a_2}{s_1s_2}) = -\frac{\alpha(a_1a_2)}{\alpha(s_1^2s_2^2)} \otimes \overline{T_\alpha(s_1s_2)} + \frac{1}{\alpha(s_1s_2)} \otimes \overline{T_\alpha(a_1a_2)} =$

$-\frac{\alpha(a_1a_2)}{\alpha(s_1^2s_2^2)} \otimes \overline{\alpha(s_1)T_\alpha(s_2) + \alpha(s_2)T_\alpha(s_1)} + \frac{1}{\alpha(s_1s_2)} \otimes \overline{\alpha(a_1)T_\alpha(a_2) + \alpha(a_2)T_\alpha(a_1)} =$

$\alpha(\frac{a_1}{s_1^2})d_\alpha(\frac{a_2}{s_2}) + \alpha(\frac{a_2}{s_2^2})d_\alpha(\frac{a_1}{s_1})$.

So there exists a unique A_S-linear map

$$h\colon I_\alpha(A_S/k)/I_\alpha(A_S/k)^2 \to A_S \otimes_A I_\alpha(A/k)/I_\alpha(A/k)^2$$

such that $d_\alpha = h \circ D_\alpha$, where $D_\alpha\colon A_S \to I_\alpha(A_S/k)/I_\alpha(A_S/k)^2$ is the composition

$$A_S \xrightarrow{T_\alpha} I_\alpha(A_S/k) \xrightarrow{\pi} I_\alpha(A_S/k)/I_\alpha(A_S/k)^2.$$

The map $\psi\colon A_S \otimes_A I_\alpha(A/k)/I_\alpha(A/k)^2 \to I_\alpha(A_S/k)/I_\alpha(A_S/k)^2$ defined by

$$\psi\left(\frac{a}{s} \otimes \overline{T_\alpha(b)}\right) = \frac{a}{s}\overline{T_\alpha(\frac{b}{1})}$$

is an inverse for h. $\square$

4. The algebra of q-difference operators and its homology

Let q be a complex number $\neq 0, 1$, and let D_q be the algebra of q-difference operators on $\mathbf{C}[x, x^{-1}]$. By definition D_q [5] is the algebra of all linear endomorphisms of $\mathbf{C}[x, x^{-1}]$ generated by multiplications by Laurent polynomials and by Jackson's q-differentiation operator ∂_q defined for any polynomial P by

$$\partial_q(P) = \frac{P(qx) - P(x)}{qx - x}.$$

As a complex associative algebra D_q is generated by x, x^{-1} and ∂_q and the relation $\partial_q x - qx\partial_q = 1$, which is the q-analogue of the Heisenberg relation for differential operators. The family $\{x^i \partial_q^j\}_{i \in \mathbf{Z}, j \in \mathbf{N}}$ is a basis of D_q. It is convenient to introduce the algebra automorphism η_q of $\mathbf{C}[x, x^{-1}]$ defined by $\eta_q(x) = qx$. Since $\eta_q = 1 + (q-1)x\partial_q$, the automorphism η_q belongs to D_q. We have the additional relations $\partial_q x - x\partial_q = \eta_q$ and $\eta_q x = qx\eta_q$. The q-differentiation operator is not a derivation, but a η_q-derivation; namely for all $P, Q \in \mathbf{C}[x, x^{-1}]$ we have

$$\partial_q(PQ) = \eta_q(P)\partial_q(Q) + \partial_q(P)Q.$$

It is easy to check that $\{x^i \partial_q\}_{i \in \mathbf{Z}}$ is a basis of the vector space $D_{\mathbf{C}}^{\eta_q}(\mathbf{C}[x, x^{-1}])$ of all η_q-derivations of $\mathbf{C}[x, x^{-1}]$.

Properties 4.1. [5] *For integers $n \in \mathbf{Z}$, set $(n)_q = 1 + q + \cdots + q^{n-1}$. Then*

a) $\partial_q^j x = q^j x \partial_q^j + (j)_q \partial_q^{j-1}$

b) $\partial_q x^i = q^i x^i \partial_q + (i)_q x^{i-1}$.

In [5], Kassel shows that the Hochschild homology groups of D_q are the homology groups of the complex

$$0 \longrightarrow D_q \otimes \wedge^2 V_q \xrightarrow{\beta_q} D_q \otimes V_q \xrightarrow{\beta_q} D_q \longrightarrow 0$$

where V_q is a two-dimensional vector space with basis $\{dx, d\partial_q\}$, and for any $M \in D_q$

$\beta_q(M\ dx \wedge d\partial_q) = (xM - qMx)d\partial_q - (q\partial_q M - M\partial_q)dx$

$\beta_q(Mdx) = xM - Mx$

$\beta_q(Md\partial_q) = \partial_q M - M\partial_q$

Let S be the commutative $\mathbf{C}$-algebra generated by x, x^{-1}, ξ, and $\alpha \colon S \to S$ the morphism of algebras defined by $\alpha(x) = qx$, $\alpha(\xi) = q\xi$. Then,

$\frac{\partial_q(x^i \xi^j)}{\partial_q(x)} = \frac{(qx)^i \xi^j - x^i \xi^j}{qx - x} = (i)_q x^{i-1} \xi^j$

$\frac{\partial_q(x^i \xi^j)}{\partial_q(\xi)} = \frac{x^i (q\xi)^j - x^i \xi^j}{q\xi - \xi} = (j)_q x^i \xi^{j-1}$.

We shall compare the complex $(D_q \otimes \bigwedge^2 V_q, \beta_q)$ with the q-de Rham complex of S

$$0 \longrightarrow \Omega_{\mathbf{C}}^{\alpha,0}(S) \xrightarrow{d_1} \Omega_{\mathbf{C}}^{\alpha,1}(S) \xrightarrow{d_2} \Omega_{\mathbf{C}}^{\alpha,2}(S) \longrightarrow 0$$

where,

$\Omega_{\mathbf{C}}^{\alpha,0}(S) = S$

$\Omega_{\mathbf{C}}^{\alpha,1}(S) = \Omega_{\mathbf{C}}^{\alpha}(S) = I_q/I_q^2, \qquad (I_q = I_\alpha(S/\mathbf{C}))$

$\Omega_{\mathbf{C}}^{\alpha,2}(S) = \Omega_{\mathbf{C}}^{\alpha}(S) \wedge \Omega_{\mathbf{C}}^{\alpha}(S),$

$\Omega_{\mathbf{C}}^{\alpha}(S)$ is generated by $dx = (1 \otimes x - qx \otimes 1) + I_q^2$ and $d\xi = (1 \otimes \xi - q\xi \otimes 1) + I_q^2$, and

$$d_1(x^i\xi^j) = \left(q\frac{\partial_q(x^i\xi^j)}{\partial_q(x)} + (q-1)\frac{\partial_q(x^{i+1}\xi^{j+1})}{\partial_q(x)}\right)dx - \left(q\frac{\partial_q(x^i\xi^j)}{\partial_q(\xi)} + (q-1)\frac{\partial_q(x^{i+1}\xi^{j+1})}{\partial_q(\xi)}\right)d\xi$$

$$d_2(x^i\xi^j dx) = \left(\frac{\partial_q(x^i\xi^j)}{\partial_q(\xi)} + (q-1)\frac{\partial_q(x^{i+1}\xi^j)}{\partial_q(\xi)}\xi\right)dx \wedge d\xi$$

$$d_2(x^i\xi^j d\xi) = \left(\frac{\partial_q(x^i\xi^j)}{\partial_q(x)} + (q-1)x\frac{\partial_q(x^i\xi^{j+1})}{\partial_q(x)}\right)dx \wedge d\xi$$

Consider the map $\sigma_*\colon D_q \otimes \bigwedge^* V_q \to \Omega_{\mathbf{C}}^{\alpha,2-*}(S)$ defined by

$\sigma_0(x^i\partial_q^j) = x^i\xi^j dx \wedge d\xi$
$\sigma_1(x^i\partial_q^j dx) = -x^i\xi^j dx$
$\sigma_1(x^i\partial_q^j d\partial_q) = x^i\xi^j d\xi$
$\sigma_2(x^i\partial_q^j dx \wedge d\partial_q) = x^i\xi^j$

Lemma 4.2. *$\sigma_*\colon D_q \otimes \bigwedge^* V_q \to \Omega_{\mathbf{C}}^{\alpha,2-*}(S)$ is a chain bijection, and induces a bijection from $H_*(D_q)$ onto $H^*_{q-DR}(S)$.*

Proof. It is obvious.

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