α-DERIVATIONS II: THE NON-COMMUTATIVE CASE

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Dedicated to the memory of our advisor and friend Orlando Eugenio Villamayor.

Abstract

We give a characterization of α-derivations of A into A-bimodules M for general algebras A and consider another category of bimodules, the α-central bimodules. We obtain a complete description of the α-derivations of the quantum plane $\mathbb{C}_q[x,y]$ and the quantum torus $\mathbb{C}_q[x,y,x^{-1},y^{-1}]$ when q is not a root of the unity. This description is given for an arbitrary algebra automorphism α.

1 Introduction

Let A be a unital k-algebra, $\alpha: A \to A$ a morphism of algebras. In [8] we studied α-derivations of A into A-bimodules M for A commutative, and presented a characterization of them, using the analogies with the classical theory of derivations. Now we give their characterization in the general case, and we consider another category of bimodules, the α-central bimodules, which is a full subcategory of the category of all bimodules. We construct for the corresponding embedding functor a left adjoint $M \to M_Z$ and a right adjoint $M \to M^Z$.

By applying the functor $M \to M_Z$ to the universal α-differential calculus on $d_\alpha : A \to \Omega_k^\alpha(A)$ we produce the universal α-differential calculus for α-central bimodules, i.e. a derivation $d_Z : A \to \Omega_k^\alpha(A)_Z$, where $\Omega_k^\alpha(A)_Z$ is an α-central bimodule such that any derivation of A in an α-central bimodule factorizes through d_Z on a unique homomorphism from $\Omega_k^\alpha(A)_Z$ in the bimodule. In the case where A is commutative, an α-central bimodule M is an A-bimodule that verifies $ma = \alpha(a)m$, for $m \in M, a \in A$ and $\Omega_k^\alpha(A)_Z$ reduces to the module of α-differentials $\Omega_k^\alpha(A) = I_\alpha/I_\alpha^2$ characterized in [8].

We also describe the derived functor of α-derivations in terms of a Hochschild homology functor, and obtain a characterization of separable algebras in terms of α-derivations.

In section 5, we obtain a complete description of the α-derivations of the quantum plane $\mathbb{C}_q[x,y]$, and the quantum torus $\mathbb{C}_q[x,y,x^{-1},y^{-1}]$ when q is not

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a root of the unity. This description is given for an arbitrary algebra automorphism α and provides a characterization of $\mathbb{C}_q[x, y]$ in two kinds. We called the first one "automorphisms of q-type" (i.e. α is such that $\alpha(x) = q^j x$, $\alpha(y) = q^{-i} y$ for some $i, j \in \mathbb{N}$). In this case we extend previous results obtained by [5] and [6], since we are able to write every α-derivation as the sum of an inner α-derivation and another α-derivation which is given explicitly in terms of α and the α-center of the algebra (i.e. twisted Hochschild cohomology in degree 0).

In the second case, if $\alpha \in \text{Aut}(\mathbb{C}_q[x, y]) \backslash \{\alpha: \alpha(x) = q^j x,\ \alpha(y) = q^{-i} y\}$, we prove that every α-derivation is inner.

These results can be easily extended for arbitrary n and relations of type $x_i x_j = q x_j x_i$, where $i \geq j + 1$. Alev and Chamarie [1] proved that if $n \neq 3$, every derivation of the quantum n-dimensional space is inner and, as a consequence, every automorphism is linear. On one hand this result shows that for $n \neq 3$ we consider every possible automorphism. On the other, we can look at Theorem 5.4 and Corollaries 5.5 and 5.6 as a generalization of the result of [1] concerning inner derivations.

2 α-Central bimodules

Let A be an associative k-algebra with 1 and $\alpha: A \to A$ a morphism of algebras. We recall that d_α is an α-derivation of A into an A-bimodule M if and only if d_α is a k-linear map, $d_\alpha: A \to M$, such that

$$d_\alpha(ab) = \alpha(a) d_\alpha(b) + d_\alpha(a) b \qquad \text{for } a, b \in A.$$

Now we describe the construction of $\Omega_k^\alpha(A)$. A and $A \otimes A^{op}$ are considered as A-bimodules with the structures defined by $a \circ x \circ b = a x \alpha(b)$, and $a \circ (x \otimes y) \circ b = ax \otimes yb$. Let $\Omega_k^\alpha(A) = I_\alpha = \text{Ker}(A \otimes A^{op} \xrightarrow{\mu_\alpha} A)$, where $\mu_\alpha(a \otimes b) = a\alpha(b)$, and $d_\alpha: A \to \Omega_k^\alpha(A)$ is defined by $d_\alpha(a) = 1 \otimes a - \alpha(a) \otimes 1$.

Proposition 2.1 *The pair $(d_\alpha, \Omega_k^\alpha(A))$ is characterized by the following universal property: Let δ be an α-derivation of A with values in an A-bimodule M, then there exists a unique homomorphism of bimodules $i_\delta: \Omega_k^\alpha(A) \to M$ such that $\delta = i_\delta \circ d_\alpha$.*

Proof: Let $\delta: A \to M$ be an α-derivation. We define $\Theta: A \otimes A^{op} \to M$ by $\Theta(a \otimes b) = a\delta(b)$. Now, consider the composition

$$i_\delta: \Omega_k^\alpha(A) \to A \otimes A^{op} \xrightarrow{\Theta} M$$

A direct computation shows that $i_\delta \circ d_\alpha(a) = \Theta(1 \otimes a - \alpha(a) \otimes 1) = \delta(a)$. Since the A-bimodule $\Omega_k^\alpha(A)$ is generated by $\{d_\alpha(a), \quad a \in A\}$, there can be only one mapping i_δ such that $i_\delta \circ d_\alpha = \delta$. Hence d_α is universal. □

Remark 2.2 *Setting $\Omega^{0,\alpha}(A) = A$, $\Omega^{1,\alpha}(A) = \Omega_k^\alpha(A)$, and $\Omega^{n,\alpha}(A) = \Omega^{1,\alpha}(A) \otimes_A \ldots \otimes_A \Omega^{1,\alpha}(A)$, $\Omega^\alpha(A) = \bigoplus \Omega^{n,\alpha}(A)$ is naturally a graded algebra, and there is a unique α-differential d_α on $\Omega^\alpha(A)$ extending the derivation $d_\alpha: \Omega^{0,\alpha}(A) \to$*

$\Omega^{1,\alpha}(A)$. The graded α-differential algebra $\Omega^\alpha(A)$ is characterized by the following universal property: Let $\phi: A \to \Omega'$ be an homomorphism of algebras with units where Ω' is a graded α-differential algebra, then there is a unique homomorphism of graded α-differential algebras $\widehat{\phi}: \Omega^\alpha(A) \to \Omega'$ which extends ϕ.

Let M be an A-bimodule. We say that M is an α-central bimodule if $ma = \alpha(a)m$ for $m \in M$, $a \in Z(A)$, the center of A. Obviously, if M is an α-central bimodule, the corresponding bimodule structure over $Z(A)$ is induced by a structure of $Z(A)$-module.

Example 2.3

i) If A is commutative, i.e. $Z(A) = A$, an α-central bimodule is an A-bimodule M such that $ma = \alpha(a)m$, for $m \in M$ and $a \in A$, (see [8]).

ii) If $Z(A) = k$, all bimodules are α-central bimodules.

Properties 2.4

1) Sub-bimodules of α-central bimodules are α-central bimodules.

2) Quotients of α-central bimodules are α-central bimodules.

3) The product of α-central bimodules is an α-central bimodule.

4) If M and N are α-central bimodules then

$$M \otimes_\alpha N = M \otimes_k N / \{ma \otimes n - m \otimes \alpha(a)n, \quad m \in M, \quad a \in Z(A)\}$$

is an α-central bimodule.

Properties (1), (2) and (3) imply that projective limits of α-central bimodules are α-central bimodules, and inductive limits of α-central bimodules are α-central bimodules.

The category of α-central A-bimodules is a full subcategory of the category of all bimodules. Let I_Z be the canonic functor that identifies the category of α-central bimodules with a full subcategory of the category of all bimodules. We shall construct covariant functors $M \to M_Z$, $M \to M^Z$ that are left and right adjoints functors of I_Z, respectively.

Let M be an A-bimodule. We define

$$M_Z = M / < ma - \alpha(a)m, \quad a \in Z(A), \quad m \in M >$$

which is an α-central bimodule, $p_Z: M \to M_Z$ the canonical projection, and

$$M^Z = \{m \in M: ma = \alpha(a)m, \quad \forall a \in Z(A)\}$$

which is a submodule of M, and is an α-central bimodule, $i^Z: M^Z \to M$ the canonical inclusion.

The following facts are immediate consequences of the previous discussion:

Proposition 2.5

i) For all morphisms of bimodules $\phi: M \to N$, *N an α-central bimodule, there exists a unique morphism of bimodules* $\phi_Z: M_Z \to N$ *such that* $\phi = \phi_Z \circ p_Z$.

ii) For all morphisms of bimodules $\psi: N \to M$, *N an α-central bimodule, there exists a unique morphism of bimodules* $\psi^Z: N \to M^Z$ *such that* $\psi = i^Z \circ \psi^Z$.

Corollary 2.6

i) The functor $M \to M_Z$ *is a left adjoint of* I_Z *and so it is right exact.*

ii) The functor $M \to M^Z$ *is a right adjoint of* I_Z *and so it is left exact.*

iii) I_Z *is a right and left exact functor.*

Remark 2.7 *The identity*

$$m \otimes na - \alpha(a)m \otimes n = m \otimes (na - \alpha(a)n) + (ma - \alpha(a)m) \otimes n + m \otimes \alpha(a)n - ma \otimes n$$

implies that if M and N are α-central bimodules, then $(M \otimes_k N)_Z = M \otimes_\alpha N$.

Remark 2.8 *The following conditions are equivalent*

i) M is an α-central bimodule

ii) $M = M_Z$

iii) $M = M^Z$

The map $d_Z: A \to \Omega^\alpha(A)_Z$ defined by the composition

$$A \xrightarrow{d_\alpha} \Omega^\alpha(A) \xrightarrow{p_Z} \Omega^\alpha(A)_Z$$

is an α-derivation.

The following Proposition and Corollary show that $\Omega_k^\alpha(A)_Z$ is the generalization of the module of Kähler differentials given for $\alpha = id$ and A commutative.

Proposition 2.9 *The pair* $(d_Z, \Omega_k^\alpha(A)_Z)$ *is characterized by the following universal property: Let δ be an α-derivation of A with values in an α-central bimodule M, then there exists a unique bimodule homomorphism* $i_\delta: \Omega_k^\alpha(A)_Z \to M$ *such that* $\delta = i_\delta \circ d_Z$.

Proof: If follows immediately by adjunction and the universal property that characterizes $(d_\alpha, \Omega_k^\alpha(A))$. □

Corollary 2.10 *If A is commutative,* $\Omega_k^\alpha(A)_Z = I_\alpha/I_\alpha^2$, *where* $I_\alpha = \mathrm{Ker}(A \otimes A \xrightarrow{\mu_\alpha} A)$.

Proof: I_α is generated by $\{T_\alpha(x) = 1 \otimes x - \alpha(x) \otimes 1, \quad x \in A\}$, and $T_\alpha(x)a - \alpha(a)T_\alpha(x) = (1 \otimes a - \alpha(a) \otimes 1)T_\alpha(x) = T_\alpha(a)T_\alpha(x)$. So, $I_\alpha(A)_Z = I_\alpha/I_\alpha^2$. □

3 The derived functor of α-derivations

This section begins with a Proposition that provides a characterization of twisted Hochschild cohomology in terms of α-derivations.

Let S denote the class of all k-split A-bimodule morphisms (morphisms which have a k-linear left or right inverse). We say that an A-bimodule I is S-injective in case, whenever a monomorphism $f: M \to M'$ is in S, the induced map

$$f^*: \mathrm{Hom}_{A\otimes A^{op}}(M', I) \to \mathrm{Hom}_{A\otimes A^{op}}(M, I)$$

is surjective. We say that a short exact sequence $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is S-exact if it is exact and $f \in S$ (or $g \in S$). Other sequences are S-exact if they are made up from short S-exact sequences in the obvious way.

It is well-known that the category of A-bimodules has enough S-injectives (Eilenberg-MacLane), and this allows us to obtain derived functors in the usual way, since we have S-exact S-injective resolutions.

Let $\mathrm{Der}_\alpha(A, M) = \{d_\alpha: A \to M, \quad d_\alpha \quad \text{an } \alpha\text{-derivation}\}$. Consider $\mathrm{Der}_\alpha(A, .)$ as a functor from A-bimodules to abelian groups, and let $H^n_\alpha(A, .)$ be its n-th derived functor (using the class S).

Proposition 3.1 $H^0_\alpha(A, M) = \mathrm{Der}_\alpha(A, M)$.

Proof: This follows at once since $\mathrm{Der}_\alpha(A, .)$ is left exact. □

Remark 3.2 *For an A-bimodule M, we consider another A-bimodule, denoted $\widetilde{M}$, with A-bimodule structure given by $a \circ m \circ b = \alpha(a)mb$. Then, there is an isomorphism*

$$\mathrm{Der}_\alpha(A, M) = \mathrm{Der}(A, \widetilde{M})$$

The functor $\mathrm{Der}(A, .)$ *is naturally equivalent to* $\mathrm{Hom}_{A\otimes A^{op}}(\Omega_k(A), .)$ *, where* $\Omega_k(A) = I = \mathrm{Ker}(A \otimes A^{op} \xrightarrow{\mu} A)$, $\mu(a \otimes b) = ab$.

For an A-bimodule M, let $\widehat{H}(M, .)$ be the derived functor (using S) of the functor $\mathrm{Hom}_{A\otimes A^{op}}(M, .)$. Then, if M is the A-bimodule A with the trivial structure $a \circ x \circ b = axb$, $\widehat{H}(A, N)$ is the usual Hochschild cohomology of A with coefficients in N.

Proposition 3.3 *Let A be the A-bimodule with the trivial structure $a \circ x \circ b = axb$, M an A-bimodule. Then, if $n \geq 1$,*

$$H^n_\alpha(A, M) = \widehat{H}^{n+1}(A, \widetilde{M})$$

Proof:

$$0 \to \Omega_k(A) \overset{\leftarrow}{\longrightarrow} A \otimes A^{op} \xrightarrow{\mu} A \to 0$$

is an S-exact sequence ($s: A \otimes A^{op} \to \Omega_k(A)$ defined by $s(a \otimes b) = a \otimes b - 1 \otimes ab$ is a k-linear left inverse of the monomorphism). The functors $\mathrm{Der}_\alpha(A, .)$, $\mathrm{Hom}_{A\otimes A^{op}}(\Omega^\alpha_k(A), .)$ and $\mathrm{Hom}_{A\otimes A^{op}}(\Omega_k(A), \tilde{.})$ are naturally equivalent (Proposition 3.3 and Remark 3.4), and the functor $M \to \widetilde{M}$ is exact.

Hence $H^*_\alpha(A,M) = \widehat{H}^*(\Omega^\alpha_k(A),M) = \widehat{H}^*(\Omega_k(A),\widetilde{M})$ and, for $n \geq 1$, we have exact sequences

$$0 = \widehat{H}^n(A\otimes A^{op},\widetilde{M}) \to \widehat{H}^n(\Omega_k(A),\widetilde{M}) \to \widehat{H}^{n+1}(A,\widetilde{M}) \to \widehat{H}^{n+1}(A\otimes A^{op},\widetilde{M}) = 0$$

□

4 The α-Hochschild cohomology

We say that $HH^*_\alpha(A,M) = \widehat{H}^*(A,\widetilde{M})$ is the α-Hochschild cohomology of the algebra A, with coefficients in the A-bimodule M.

Put $\overline{A} = A/k$, and consider the standard normalized Hochschild resolution of A

$$C_* : \ldots \to A\otimes \overline{A}^{\otimes 2}\otimes A \xrightarrow{b'} A\otimes\overline{A}\otimes A \xrightarrow{b'} A\otimes A \xrightarrow{\mu} A \to 0$$

$$b'(a_0\otimes\ldots a_n) = \sum_{i=0}^{n-1}(-1)^i a_0\otimes\ldots\otimes a_i a_{i+1}\otimes\ldots\otimes a_n$$

Proposition 4.1 *The complex C_* is an S-projective resolution of the A-bimodule A (with trivial structure).*

Proof: The k-linear map

$$s: A\otimes\overline{A}^{\otimes n-2}\otimes A \to A\otimes\overline{A}^{\otimes n-1}\otimes A, \quad s(a_1\otimes\ldots\otimes a_n) = (-1)^{n+1}1\otimes a_1\otimes\ldots\otimes a_n$$

satisfies the formula $b's + sb' = id$, showing that the complex is S-exact. □

Corollary 4.2

$$HH^0_\alpha(A,M) = \{m\in M : ma = \alpha(a)m\}$$

and

$$HH^1_\alpha(A,M) = \mathrm{Der}_\alpha(A,M)/\,\mathrm{Int}_\alpha(A,M)$$

where $\mathrm{Int}_\alpha(A,M) = \{f\in \mathrm{Der}_\alpha(A,M) : f(a) = \alpha(a)m - ma\}$ *is the submodule of inner α-derivations.*

Proof: Notice that the complex is given, in degree n, by:

$$\begin{aligned}\mathrm{Hom}_{A\otimes A^{op}}(A\otimes\overline{A}^{\otimes n-2}\otimes A,\widetilde{M}) &= \mathrm{Hom}_{A\otimes A^{op}}((A\otimes A^{op})\otimes\overline{A}^{\otimes n-2},\widetilde{M})\\ &= \mathrm{Hom}_k(\overline{A}^{\otimes n-2},\widetilde{M})\end{aligned}$$

so we have that the α-Hochschild cohomology $HH^*_\alpha(A,M)$ is the homology of the complex

$$0\to\widetilde{M}\xrightarrow{b}\mathrm{Hom}_k(\overline{A},\widetilde{M})\xrightarrow{b}\mathrm{Hom}_k(\overline{A}\otimes\overline{A},\widetilde{M})\xrightarrow{b}\ldots$$

where $b: \mathrm{Hom}_k(\overline{A}^{\otimes n},\widetilde{M})\to\mathrm{Hom}_k(\overline{A}^{\otimes n+1},\widetilde{M})$ is given by

$$
\begin{aligned}
b(f)(a_0 \otimes \ldots \otimes a_n) &= a_0 \circ f(a_1 \otimes \ldots \otimes a_n) \\
&+ \sum_{i=1}^{n-1} (-1)^i f(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n) \\
&+ (-1)^n f(a_0 \otimes \ldots \otimes a_{n-1}) \circ a_n \\
&= \alpha(a_0) f(a_1 \otimes \ldots \otimes a_n) \\
&+ \sum_{i=1}^{n-1} (-1)^i f(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n) \\
&+ (-1)^n f(a_0 \otimes \ldots \otimes a_{n-1}) a_n
\end{aligned}
$$

Then,

$$
\begin{aligned}
HH^0_\alpha(A, M) &= \mathrm{Hom}_{A \otimes A^{op}}(A, \widetilde{M}) \\
&= \{m \in \widetilde{M} : m \circ a = a \circ m\} = \{m \in M : ma = \alpha(a) m\}
\end{aligned}
$$

and

$$
HH^1_\alpha(A, M) = \mathrm{Der}(A, \widetilde{M}) / \mathrm{Int}(A, \widetilde{M}) = \mathrm{Der}_\alpha(A, M) / \mathrm{Int}_\alpha(A, M)
$$

□

Proposition 4.3 *The following facts are equivalent:*

i) A is separable (see [2]),

ii) any derivation of A with values in a bimodule is inner,

iii) any α-derivation of A with values in a bimodule is inner, for any automorphism α.

Proof: It is well known that the first and the second assertions are equivalent. The proof of the previous corollary shows the equivalence with the third one.

Remark 4.4 *If A is commutative, $HH^0_\alpha(A, M) = M^Z$. If A is commutative and M is an α-central A-bimodule, $HH^1_\alpha(A, M) = \mathrm{Der}_\alpha(A, M)$.*

5 α-derivations of the quantum plane

This section is devoted to the computation of α-derivations of the quantum plane and the quantum torus, for an arbitrary algebra automorphism α. It appears that there are two kinds of automorphisms: those of q-type (sending $q \mapsto q^i x$ and $y \mapsto q^{-j} y$), and the other ones. If α is not of q-type, then every α-derivation is inner (Theorem 5.4 and Corollary 5.5). If not, then every α-derivation can be written as the sum of an inner $\alpha-$derivation and another one (which is given explicitly).

Let $q \in \mathbb{C}$ be a fixed nonzero complex number, non-root of unity ($q^n \neq 1$ for any $n \in \mathbb{N}$). Let $\mathbb{C}_q[x, y]$ be the skew polynomial ring, called the quantum plane, where $yx = qxy$. Note that the monomials of the form $x^i y^j$ form a vector space basis of $\mathbb{C}_q[x, y]$, and it is easy to see that

$$\begin{aligned}(x^i y^j)(x^k y^s) &= q^{jk} x^{i+k} y^{j+s} \\ (x^i y^j)(x^k y^s) &= q^{jk-is} (x^k y^s)(x^i y^j)\end{aligned}$$

We will compute the α-derivations of the quantum plane, for α an automorphism of $\mathbb{C}_q[x, y]$.

Lemma 5.1 *[1, Prop. 1.4.4] If α is a $\mathbb{C}$ automorphism of $\mathbb{C}_q[x, y]$, it must take $x^i y^j$ to $a^i b^j x^i y^j$, with $a, b \in \mathbb{C} \backslash \{0\}$.*

Let $\mathbb{C}_q[x, y, x^{-1}, y^{-1}]$ be the skew Laurent polynomial ring, the quantum torus, sometimes called the multiplicative analog of the Weyl algebra.

Proposition 5.2

i) Any automorphism of $\mathbb{C}_q[x, y]$ extends uniquely to an automorphism of $\mathbb{C}_q[x, y, x^{-1}, y^{-1}]$.

ii) Let α be an automorphism of $\mathbb{C}_q[x, y]$, any α-derivation of $\mathbb{C}_q[x, y]$ extends uniquely to an α-derivation of $\mathbb{C}_q[x, y, x^{-1}, y^{-1}]$.

Proof: We define $\alpha(x^{-1}) = \alpha(x)^{-1}$ and $\alpha(y^{-1}) = \alpha(y)^{-1}$ for part (1), and $d_\alpha(x^{-i}) = -\alpha(x)^{-i} d_\alpha(x^i) x^{-i}$ and $d_\alpha(y^{-i}) = -\alpha(y)^{-i} d_\alpha(y^i) y^{-i}$ for part (2).
The uniqueness follows from the fact that $\alpha(1) = 1$ and $d_\alpha(1) = 0$. □

Now we set

$$Z^\alpha(\{x^i y^j\}_{i,j \in \mathbb{Z}}) = \{x^k y^s : (x^k y^s)(x^i y^j) = \alpha(x^i y^j)(x^k y^s), \quad \text{for all} \quad i, j \in \mathbb{Z}\}$$

Lemma 5.3 *$Z^\alpha(\{x^i y^j\}_{i,j}) \neq \emptyset$ if and only if there exist $i_0, j_0 \in \mathbb{Z}$ such that $\alpha(x) = q^{j_0} x$, $\alpha(y) = q^{-i_0} y$. In this case, $Z^\alpha(\{x^i y^j\}_{i,j}) = \{x^{i_0} y^{j_0}\}$.*

Proof: $Z^\alpha(\{x^i y^j\}_{i,j}) \neq \emptyset$ if there exists $x^{i_0} y^{j_0}$ such that

$$(x^{i_0} y^{j_0})(x^i y^j) = \alpha(x^i y^j)(x^{i_0} y^{j_0}) \qquad \text{for all } i, j \in \mathbb{Z}$$

Setting $(i, j) = (1, 0)$ we get

$$x^{i_0} y^{j_0} x = \alpha(x) x^{i_0} y^{j_0}$$

So $\alpha(x) = q^{j_0} x$. Similarly, setting $(i, j) = (0, 1)$, we get $\alpha(y) = q^{-i_0} y$.
On the other hand, set $\alpha(x) = q^{j_0} x$, $\alpha(y) = q^{-i_0} y$. If $x^r y^s \in Z^\alpha(\{x^i y^j\}_{i,j})$, then

$$(x^r y^s)(x^i y^j) = \alpha(x^i y^j)(x^r y^s) = q^{ij_0 - ji_0}(x^i y^j)(x^r y^s) \qquad \text{for all } i, j \in \mathbb{Z}$$

Taking $(i, j) = (1, 0)$, we have $s = j_0$; and taking $(i, j) = (0, 1)$, we get $r = i_0$.□

Theorem 5.4 *Let* $\alpha \in \mathrm{Aut}(\mathbb{C}_q[x,y])\backslash\{\alpha: \alpha(x) = q^j x, \quad \alpha(y) = q^{-i} y\}$. *Then any* α*-derivation of* $\mathbb{C}_q[x,y,x^{-1},y^{-1}]$ *is inner.*

Proof: Lemma 5.1 implies that $\alpha(x^i y^j) = a^i b^j x^i y^j$. We will denote $A(x^i y^j) = a^i b^j$. Let d_α be an α-derivation of $\mathbb{C}_q[x,y,x^{-1},y^{-1}]$. Note that $\{x^i y^j\}_{i,j\in\mathbb{Z}}$ is a vector space basis of $\mathbb{C}_q[x,y,x^{-1},y^{-1}]$. For each $z \in \{x^i y^j\}$, write

$$d_\alpha(z)z^{-1} = \sum_{i,j} a_{ij}(z) x^i y^j$$

Then, $d_\alpha(z) = \sum_{i,j} a_{ij}(z) x^i y^j z$, and for each z only finitely many $a_{ij}(z)$ can be nonzero.

Let $z = x^k y^s$, $w = x^r y^l$, $u = x^{k+r} y^{s+l}$. Since $zw = q^{sr} u$ and $z(x^i y^j) = q^{si-kj}(x^i y^j)z$, we have

$$\begin{aligned} d_\alpha(zw) &= q^{sr} d_\alpha(u) = q^{sr} \sum a_{ij}(u) x^i y^j u \\ &= \sum a_{ij}(u) x^i y^j zw \end{aligned}$$

and

$$\begin{aligned} d_\alpha(zw) &= \alpha(z) d_\alpha(w) + d_\alpha(z) w \\ &= A(z) z \sum a_{ij}(w) x^i y^j w + \sum a_{ij}(z) x^i y^j zw \\ &= \sum (q^{si-kj} A(z) a_{ij}(w) + a_{ij}(z)) x^i y^j zw \end{aligned}$$

Equating coefficients gives,

$$a_{ij}(u) = q^{si-kj} A(z) a_{ij}(w) + a_{ij}(z)$$

But,

$$\begin{aligned} q^{si-kj} A(z) a_{ij}(w) + a_{ij}(z) &= a_{ij}(u) = a_{ij}(x^{k+r} y^{s+l}) \\ &= a_{ij}(x^{r+k} y^{l+s}) \\ &= q^{li-rj} A(w) a_{ij}(z) + a_{ij}(w) \end{aligned}$$

So we obtain

$$(q^{si-kj} A(z) - 1) a_{ij}(w) = (q^{li-rj} A(w) - 1) a_{ij}(z)$$

Now, $q^{si-kj} A(z) = 1$ if and only if $(x^i y^j) z = \alpha(z)(x^i y^j)$. Let

$$V_{ij} = \{z = x^k y^s : q^{si-kj} A(z) = q^{si-kj} a^k b^s \neq 1\}$$

We know that $V_{ij} \neq \emptyset$ because $(a,b) \neq (q^j, q^{-i})$. Let $z, w \in V_{ij}$. Then

$$\frac{a_{ij}(w)}{q^{li-rj} A(w) - 1} = \frac{a_{ij}(z)}{q^{si-kj} A(z) - 1}$$

and thus these factors depend only upon i, j. Writing c_{ij} for this common value, we have $a_{ij}(z) = c_{ij}(q^{si-kj}A(z) - 1)$ for all $z \in V_{ij}$. Furthermore, if $z, w \in V_{ij}$, $zw = q^{sr}u$, then

$$\begin{aligned} a_{ij}(u) &= q^{si-kj}A(z)a_{ij}(w) + a_{ij}(z) \\ &= q^{si-kj}A(z)c_{ij}(q^{li-rj}A(w) - 1) + c_{ij}(q^{si-kj}A(z) - 1) \\ &= c_{ij}(q^{(s+l)i-(k+r)j}A(z)A(w) - 1) \\ &= c_{ij}(q^{(s+l)i-(k+r)j}A(u) - 1) \end{aligned}$$

Let $z = x^k y^s \notin V_{ij}$. Then $q^{si-kj}a^k b^s = 1$. We want to show that z is a product of two elements of V_{ij}. It is easy to see that x, $x^{k-1}y^s \notin V_{ij}$ implies $a = q^j$, and $x^k y^{s-1}$, $y \notin V_{ij}$ implies $b = q^{-i}$, a contradiction. Therefore, we conclude that

$$a_{ij}(x^k y^s) = c_{ij}(q^{si-kj}A(x^k y^s) - 1) \qquad \text{for all } k, s \in \mathbb{Z}$$

Let $\gamma = \sum_{i,j} c_{ij}x^i y^j$. It is immediate that only finitely many c_{ij} can be nonzero, because $a_{ij}(x) = c_{ij}(q^{-j}a - 1)$ and finitely many $a_{ij}(x)$ can be nonzero. Now,

$$\begin{aligned} d_\alpha(z) = d_\alpha(x^k y^s) &= \sum_{i,j} a_{ij}(z)x^i y^j z \\ &= \sum_{i,j} c_{ij}(q^{si-kj}A(z) - 1)x^i y^j z \\ &= \sum_{i,j} c_{ij}(A(z)zx^i y^j - x^i y^j z) \\ &= \alpha(z)\gamma - \gamma z \end{aligned}$$

We have then proved that d_α is inner. □

Corollary 5.5 *Let $\alpha \in \operatorname{Aut}(\mathbb{C}_q[x, y]) \backslash \{\alpha\colon \alpha(x) = q^j x, \quad \alpha(y) = q^{-i}y\}$, $\alpha(x) \neq q^{-1}x$, $\alpha(y) \neq qy$. Then any α-derivation of $\mathbb{C}_q[x, y]$ is inner.*

Proof: Let d_α be an α-derivation of $\mathbb{C}_q[x, y]$. If we consider the extension of d_α to an α-derivation of $\mathbb{C}_q[x, y, x^{-1}, y^{-1}]$, we know that $d_\alpha(z) = \alpha(z)\gamma - \gamma z$, for $\gamma = \sum_{i,j} c_{ij}x^i y^j$. Our task is to show that $\gamma \in \mathbb{C}_q[x, y]$.

Let $z = x^k y^s \in \mathbb{C}_q[x, y]$. Then

$$\begin{aligned} d_\alpha(z) = d_\alpha(x^k y^s) &= \sum_{i,j \in \mathbb{Z}} a_{ij}(z)x^i y^j z \\ &= \sum_{i,j \in \mathbb{Z}} c_{ij}(q^{si-kj}A(z) - 1)x^i y^j z \end{aligned}$$

In particular $a_{ij}(x) = c_{ij}(q^{-j}A(x) - 1) = 0$ if $i < -1$ or $j < 0$ and $a_{ij}(y) = c_{ij}(q^i A(y) - 1) = 0$ if $i < 0$ or $j < -1$. If $A(x) = a \neq q^j$ and $A(y) = b \neq q^{-i}$, then

$$c_{ij} = \frac{a_{ij}(x)}{q^{-j}A(x) - 1} = \frac{a_{ij}(y)}{q^i A(y) - 1} = 0$$

if $i < 0$ or $j < 0$ and we are done.

If $A(x) = q^j$, then $j \neq -1$ and $A(y) \neq q^{-i}$, so

$$c_{ij} = \frac{a_{ij}(y)}{q^i A(y) - 1} = 0$$

if $i < 0$ or $j < -1$ and

$$c_{i(-1)} = \frac{a_{i(-1)}(x)}{qA(x) - 1} = 0$$

for all i.

If $A(y) = q^{-i}$ the proof follows in an analogous way. □

Corollary 5.6 *If* $\alpha \in \mathrm{Aut}(\mathbb{C}_q[x,y]) \backslash \{\alpha: \alpha(x) = q^j x, \quad \alpha(y) = q^{-i} y\}$, $\alpha(x) \neq q^{-1}x$, $\alpha(y) \neq qy$, *then:*

$$HH^0_\alpha(\mathbb{C}_q[x,y]) = HH^1_\alpha(\mathbb{C}_q[x,y]) = 0$$

.

Finally, we consider α an automorphism such that:

i) $\alpha(x) = q^{-1}x$ and $\alpha(y) \neq q^{-i}y$;

ii) $\alpha(y) = qy$ and $\alpha(x) \neq q^j x$;

iii) $\alpha(x) = q^{j_0}x$, $\alpha(y) = q^{-i_0}y$.

In the first case any α-derivation $d_\alpha : \mathbb{C}_q[x,y] \to \mathbb{C}_q[x,y]$ is given by

$$d_\alpha(x^k y^s) = \gamma'(x^k y^s) - (x^k y^s)\gamma' + \sum_{i \geq 0} c_i (q^{is} A(y)^s - 1) x^i y^{-1} (x^k y^s)$$

with $\gamma' = \sum_{i \geq 0, j \geq 0} c_{ij} x^i y^j$.

The second case is analogous to the first one.

Let us consider the last case. For $(i,j) \neq (i_0, j_0)$ we can repeat the proof above to find $\gamma = \sum_{(i,j) \neq (i_0, j_0)} c_{ij} x^i y^j$ such that

$$\begin{aligned} a_{ij}(x^k y^s) &= c_{ij}(q^{si-kj} A(x^k y^s) - 1) \\ &= c_{ij}(q^{si-kj} q^{kj_0 - si_0} - 1) \qquad \text{for all } k, s \in \mathbb{Z} \end{aligned}$$

Then any α-derivation $d_\alpha : \mathbb{C}_q[x,y,x^{-1},y^{-1}] \to \mathbb{C}_q[x,y,x^{-1},y^{-1}]$ can be written us $d_\alpha = d^{(1)}_\alpha + d^{(2)}_\alpha$, with

$$\begin{aligned} d^{(1)}_\alpha(z) &= \alpha(z)\gamma - \gamma z \\ d^{(2)}_\alpha(z) &= a_{i_0 j_0}(z) x^{i_0} y^{j_0} z \end{aligned}$$

The last α-derivation is called central because $x^{i_0} y^{j_0} \in Z^\alpha(\{x^i y^j\}_{i,j})$. Observe that $a_{i_0 j_0}$ depends multiplicatively on z, i.e. $a_{i_0 j_0}(zw) = a_{i_0 j_0}(z) + a_{i_0 j_0}(w)$.

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