Hypergeometric functions with integer homogeneities

Alicia Dickenstein

Abstract. We survey several results on A-hypergeometric systems of linear partial differential equations introduced by Gelfand, Kapranov and Zelevinsky in the case of integer (and thus resonant) parameters, in particular, those differential systems related to sparse systems of polynomial equations. We also study in particular the case of A-hypergeometric systems for which $\ker A$ has rank 1. This allows us to clarify the combinatorial meaning of the parameters in one variable classical generalized hypergeometric functions ${}_pF_{p-1}$, and to describe all such rational functions.

1. Hypergeometric functions

Given three complex parameters α, β, γ such that $\gamma \notin \mathbb{Z}_{\leq 0}$ (or if $\gamma \in \mathbb{Z}_{\leq 0}$, then $\alpha - \gamma \in \mathbb{Z}_{\geq 1}$), Gauss hypergeometric function $F(\alpha, \beta, \gamma; x)$ was introduced by Gauss in 1812 ([**15**]). For any natural number n, let $(\alpha)_n$ denote the Pochammer symbol

$$(\alpha)_n = \alpha \cdot (\alpha + 1) \dots (\alpha + n - 1).$$

Note that $(1)_n = n!$. Then, define

$$F(\alpha, \beta, \gamma; x) = \sum_{n \geq 0} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{x^n}{n!}\ ,\ |x| < 1. \tag{1.1}$$

In particular,

$$F(\alpha, \beta, \beta.x) = (1-x)^{-\alpha} \qquad - xF(1,1,2;x) = \log(1-x).$$

These functions were studied by many mathematicians including Riemann ([**29**]) who concentrated in their behaviour as functions of a complex variable, and studied its analytic continuation regarding it as a solution to the differential equation

$$x(1-x)y'' + (\gamma - (\alpha + \beta + 1)x)y' - \alpha\beta y = 0, \tag{1.2}$$

or, multiplying equation (1.2) by x and denoting $\Theta := x\frac{d}{dx}$,

$$[\Theta(\Theta + \gamma - 1) - x(\Theta + \alpha)(\Theta + \beta)](y) = 0. \tag{1.3}$$

Equation (1.2) (or (1.3)) has three regular singular points at $0, 1$ and ∞ , and it is up to normalization the general form of a second order linear differential equation with this behaviour.

1991 *Mathematics Subject Classification.* 33C70,33C20,32A27.

Partially supported by UBACYT, ANPCYT and CONICET, Argentina.

Another important feature is the following. Denote

$$A_n := \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!},$$

i.e.

$$F(\alpha,\beta,\gamma;x) = \sum_{n\geq 0} A_n x^n.$$

Then, A_{n+1}/A_n is a rational function of n, namely

$$\frac{A_{n+1}}{A_n} = \frac{(\alpha+n)(\beta+n)}{\gamma+n)(1+n)}. \tag{1.4}$$

Or, the coefficients A_n satisfy the following linear recurrence

$$(\gamma+n)(1+n)A_{n+1} \, - \, (\alpha+n)(\beta+n)A_n = 0. \tag{1.5}$$

Indeed, (1.5) is equivalent to the fact that $F(\alpha,\beta,\gamma;x)$ satisfies equation (1.3).

Gauss hypergeometric functions also have interesting integral representations and satisfy the so called contiguity relations, which express the relations among the hypergeometric series associated with shifted parameters (cf. for example [**3**] and the references therein).

The generalized (univariate) hypergeometric functions are defined as follows (cf.[**31**], [**5**]). Fix a natural number p and let $(\mu;\mu') = (\mu_1,\dots,\mu_p;\nu_1,\dots,\mu'_{p-1}) \in \mathbf{C}^{2p-1}$ where no μ'_i is a negative integer. We define

$${}_pF_{p-1}((\mu;\mu'),x) := \sum_{n=0}^{\infty} \frac{(\mu_1)_n \dots (\mu_p)_n}{(\mu'_1)_n \dots (\mu'_{p-1})_n} \frac{x^n}{n!}, \ |x|<1. \tag{1.6}$$

This definition makes also sense for negative integer values of the μ_i provided that for such i there exists an index j such that $\mu_j - \mu_i \in \mathbb{Z}_{\geq 1}$. With this terminology, Gauss function is written as ${}_2F_1((\alpha,\beta;\gamma),x)$. If we write

$${}_pF_{p-1}((\mu;\mu'),x) := \sum_{n=0}^{\infty} A_n x^n,$$

the coefficients A_n satisfy the linear recursion

$$(\mu'_1+n)\dots(\mu'_{p-1}+n)(1+n)A_{n+1} - (\mu_1+n)\dots(\mu_p+n)A_n = 0 \tag{1.7}$$

which means that ${}_pF_{p-1}((\mu;\mu'),x)$ satisfies the following linear differential equation of order p

$$[\Theta(\Theta+\mu'_1-1)\dots(\Theta+\mu'_{p-1}-1) - x(\Theta+\mu_1)\dots(\Theta+\mu_p)](y) = 0. \tag{1.8}$$

In fact, when the parameters are non resonant, it is possible to find a basis of (multivalued) solutions to (1.8) around the origin given by p functions of the form a monomial times a suitable ${}_pF_{p-1}$.

There exist many definitions of multivariate hypergeometric functions proposed by Horn, Appell, Aomoto, etc. We recall the quite general definition proposed by Gelfand and collaborators.

The A-hypergeometric (or GKZ) system of differential equations were introduced in a series of papers in the 1980's by the Gel'fand school, particularly Gel'fand, Kapranov, and Zelevinsky ([**16**], [**17**], [**18**], see also [**1**], [**30**]). They provide a multivariate generalization of the classical hypergeometric differential equation (1.2).

Let $A = \{\nu_1, \dots, \nu_s\} \subset \mathbb{Z}^d$ be a finite subset which spans a d-dimensional lattice in $\mathbb{Z}^d$. Suppose, moreover, that there exists a vector $\lambda \in \mathbb{Q}^d$ such that

$$\langle \lambda, \nu_j \rangle = 1 \ \text{ for all } j = 1, \dots, s, \tag{1.9}$$

i.e. the set A lies in a rational hyperplane which does not contain the origin. Let A also denote the $d \times s$ matrix whose columns are the vectors ν_j. Let $\mathcal{L}_A \subset \mathbb{Z}^s$ be the sublattice of elements $v \in \mathbb{Z}^s$ such that $A \cdot v = 0$. Note that A has rank d and thus $\mathcal{L}_A$ has rank $s - d$. For any $v \in \mathcal{L}_A$, denote $\mathcal{D}_v$ the differential operator in $\mathbb{C}^s$:

$$\mathcal{D}_v \ := \ \prod_{v_j > 0} (\partial_j)^{v_j} - \prod_{v_k < 0} (\partial_k)^{-v_k} \ .$$

The *A-hypergeometric system $H_A(\beta)$ with parameter* $\beta \in \mathbb{C}^{(d)}$ is defined by the following constant coefficients operators

$$\mathcal{D}_v \varphi \ = \ 0 \quad ; \quad v \in \mathcal{L}_A \tag{1.10}$$

and the following Euler operators describing A-homogeneity in infinitesimal terms

$$\sum_{j=1}^{s} \nu_{ji}\, x_j \partial_j \ \varphi \ = \ \beta_i \, \varphi \quad ; \quad i = 1, \dots, d. \tag{1.11}$$

A function $f(x_1, \dots, x_s)$, holomorphic in an open set $U \subset \mathbb{C}^s$, annihilated by this system is said to be *A-hypergeometric with homogeneity parameter β*.

In fact, one can replace equations (1.10) by a finite number of equations in an algorithmic way (cf. [**35**], Ch. 4).

Example 1.1. **Gauss system**
Consider the Gauss configuration in $\mathbb{R}^3$

$$A \ = \ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} . \tag{1.12}$$

In this case, $\mathcal{L}_A$ is generated by the vector $(1, 1, -1, -1)$ and the hypergeometric system can be reduced to the following equations on four variables x_1, x_2, x_3, x_4.

$$\begin{array}{rcl} (\partial_1\partial_2 - \partial_3\partial_4)\,(\varphi) & = & 0 \\ (x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4)\,(\varphi) & = & \beta_1\varphi \\ (x_2\partial_2 + x_3\partial_3)\,(\varphi) & = & \beta_2\varphi \\ (x_2\partial_2 + x_4\partial_4)\,(\varphi) & = & \beta_3\varphi \end{array}$$

It is easy to check for example that $1/\Delta$ is a rational solution with homogeneity $(-2, -1, -1)$, where $\Delta = 1/x_1x_2 - x_3x_4$ is the discriminant of the configuration A. This matrix A and the corresponding system (1.1) are a nice encoding for Gauss equation (1.3). In fact, given any homogeneity β and $v \in \mathbb{C}^n$ such that $A \cdot v = \beta$ and $v_1 = 0$, any solution φ of (1.1) can be written as

$$\varphi(x) = x^v \ f\left(\frac{x_1x_2}{x_3x_4}\right),$$

where $f(z)$ satisfies Gauss equation (1.3) for $\alpha = v_2, \beta = v_3, \gamma = v_4 + 1$. The general statement for this assertion is given below in Proposition 2.1.

The A-hypergeometric system is holonomic (cf. [**17**],[**1**]). For generic parameter vectors β, the holonomic rank $r_A(\beta)$, i.e. the dimension of the space of local A-hypergeometric functions at a generic point in $\mathbb{C}^s$, equals the normalized volume $\mathrm{vol}(A)$ of the convex hull $\mathrm{conv}\{\nu_1, \ldots, \nu_s\}$ of the configuration A (see 1.16 below) and it is possible to construct bases of local holomorphic solutions in terms of Γ-*series* based on triangulations of the convex hull of A. ([**17**]). The normalized volume of the configuration (1.12) is 2! times the standard volume, i.e. it is equal to 2 (so that each simplex in the two triangulations of A is assigned volume 1).

When dealing with solutions that may be developed as a Laurent series, the parameter vectors have integer coordinates and are, therefore, automatically resonant. In this case, Euler integrals and Γ-series may not give a complete system of solutions of the A-hypergeometric equations. Integer parameters correspond to solutions of interest in toric mirror symmetry ([**4**],[**23**],[**33**]). Also, as we will show in the next example and in section 3, integer parameters come into the picture when dealing with roots (and residues) associated with sparse polynomial systems.

Example 1.2. **Systems associated with toric curves**
Given coprime integers $0 < k_1 < \ldots < k_m < n$, set

$$A = \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ 0 & k_1 & \ldots & k_m & n \end{pmatrix}, \tag{1.13}$$

and $\beta = (0, -1)$. Then, the local roots $\rho(x)$ of the generic sparse polynomial

$$f(x;t) := x_0 + x_{k_1} t^{k_1} + \cdots + x_{k_m} t^{k_m} + x_n t^n ,$$

viewed as functions of the coefficients, are algebraic solutions to the A-hypergeometric system. This fact was observed for the generic univariate polynomial of degree n (i.e., $m = n-1$ and $k_i = i$ for all i) by Birkeland [**6**] and Mayr [**24**]. More recently, Sturmfels [**36**] refined their results and gave very explicit formulas for the roots in terms of Γ-series in open sets associated to any triangulation (partition) of the segment $[0, \ldots, n]$. It is not difficult to see that these formulas may be restricted to the sparse case for any open set associated with a partition with ending points contained in $0, k_1, \ldots, k_m, n$. For any integer homogeneity, all solutions to the A-hypergeometric system associated with the matrix (1.13) may be described in terms of linear combinations of the local roots, their powers, derivatives and logarithms [**10**]. In the case of the generic univariate polynomial of degree n, the constant coefficients operators may be replaced by the finite set of order two operators

$$(\partial_i \partial_j - \partial_k \partial_\ell)(\varphi) = 0, \qquad \text{for all } i + j = k + \ell.$$

For a sparse polynomial, higher order operators are needed. Consider for example the system associated with the matrix (1.13), in case the second row is $(0, 1, 3, 4)$. Then, the operators (1.10) may be replaced by the following four operators (we name the variables x_0, x_1, x_3, x_4 according to the powers of t).

$$\begin{aligned} \left(\partial_0^2 \partial_3 - \partial_1^3\right)(\varphi) &= 0 \\ \left(\partial_0^3 \partial_4 - \partial_1^4\right)(\varphi) &= 0 \\ \left(\partial_0 \partial_4 - \partial_1 \partial_3\right)(\varphi) &= 0 \\ \left(\partial_1 \partial_4^2 - \partial_3^3\right)(\varphi) &= 0. \end{aligned}$$

An implementation to get these equations is available in the computer system KAN [**37**] by N. Takayama.

Also, the powers $\rho^s(x)$, $s \in \mathbb{Z}$, of the roots of $f(x;t)$, are A-hypergeometric system with parameter $(0, -s)$. The total sum

$$p_s(x) \ := \ \rho_1^s(x) + \cdots + \rho_d^s(x)$$

is then a rational solution with the same homogeneity. Similarly, one can show that the local residues

$$\mathrm{Res}_{\rho(x)} \left(\frac{t^b}{f^a(x;t)} \frac{dt}{t} \right) \ ; \quad a, b \in \mathbb{Z}, \ a \geq 1 \tag{1.14}$$

give algebraic solutions with homogeneity $(-a, -b)$ and, again, the total sum of residues is a rational solution.

Mayr also tried to find differential equations satisfied for the roots of bivariate polynomials, but he did not find a general pattern. Also, Norguet [**25**], [**26**] found formulas for the common roots of a bivariate system of polynomials as functions of the coefficients of the given polynomials, when there are two "leading coefficients" on each polynomial, in the sense that all the remaining coefficients have enough smaller absolute value. But there is no nice known system of linear P.D.E. satisfied by the roots. However, there is a natural A-hypergeometric system associated with the (local or global) residues, from which the roots can be derived. We pursue this discusion in section §3.

The hypothesis (1.9) means that the toric ideal

$$I_A \ := \ \langle \, \xi^u - \xi^v \ : \ A \cdot u = A \cdot v \, \rangle \quad \subset \quad \mathbb{C}[\xi_1, \ldots, \xi_s] \tag{1.15}$$

is homogeneous with respect to total degree and defines a projective toric variety $X_A \subset \mathbb{P}^{s-1}$. This condition ensures that the system $H_A(\beta)$ defined by (1.10) and (1.11) has only regular singularities ([**17**], [**30**, Theorem 2.4.11]).

If I_A is Cohen-Macaulay or β is generic then

$$r_A(\beta) \quad = \quad \mathrm{vol}(A)), \tag{1.16}$$

The inequality $r_A(\beta) \geq \mathrm{vol}(A))$ holds without any assumptions on A and β. See [**1**], [**17**], [**30**] for proofs and details. If $d = 2$, i.e. when X_A is a curve, then (1.16) holds for all $\beta \in \mathbb{C}^2$ if and only if I_A is Cohen-Macaulay [**10**]. It is interesting to point out that we show that for a given (integer) homogeneity β the holonomic rank is bigger than the normalized volume $n =$ degree $(f) =$ degree(X_A) precisely when the dimension of the rational solutions increases.

The irreducible components of $\mathrm{Sing}(H_A(\beta))$ are the hypersurfaces defined by the A'-discriminants $\Delta_{A'}$, where A' runs over all *facial subsets* A, or, equivalently, $X_{A'}$ runs over the closures of torus orbits on X_A. The *A-discriminant* Δ_A is the irreducible polynomial defining the dual variety of the toric variety X_A, with the convention $\Delta_A = 1$ if that dual variety is not a hypersurface; see [**17**], [**20**], [**1**]. Then, the denominator of any rational A-hypergeometric function is a product of powers of some $\Delta_{A'}$.

In the case of toric curves, i.e. if $d = 2$ the denominator of a rational A-hypergeometric function may thus in principle contain powers of x_0, x_n, and the discriminant $\Delta(f)$. However, we show in [**10**] that there are no rational solutions whose denominator involves $\Delta(f)$ and therefore, every rational solution is in fact a Laurent polynomial. This may be a somewhat surprising result peculiar to the case $d = 2$, in view of example 1.1. The main result in [**13**] implies that, on the contrary, the Gauss system is a very particular case of a Cayley configuration (cf. Definition 3.2) and that for general configurations A all rational A-hypergeometric functions are Laurent polynomials .

The key building block for this result is the characterization of those matrices A with one dimensional kernel, i.e. of codimension one, for which there exist rational non Laurent polynomial A-hypergeometric functions. This is closely related to the study of generalized univariate hypergeometric functions (cf. [**17**]). We stress this connection in section §2.

On the other side, we prove in [**13**] that all essential Cayley configurations have rational non Laurent polynomial solutions, by means of toric residues ([**14**], [**9**], [**8**]). An important feature is our description of the denominator of the toric residues of Laurent polynomials in [**11**], which we extend to the case of toric residues of rational functions in general in section §3.

2. Codimension one systems and univariate hypergeometric functions

In this section we will study the particular case of configurations for which the codimension $n = s - d$ is equal to one. The lattice $\mathcal{L}_A$ is generated by a single vector

$$b \;=\; (b_1, \dots , b_s) \;\in\; \mathbb{Z}^s \,, \tag{2.1}$$

which is unique up to sign. By the regularity assumption (1.9), we may suppose that the first row of A is given by the vector $(1, \dots , 1)$, which implies $b_1 + \cdots + b_s = 0$, or

$$\sum_{b_j>0} b_j \;=\; -\sum_{b_k<0} b_k \;:=\; p\,. \tag{2.2}$$

Moreover, in this case

$$p \;=\; \mathrm{vol}(A) \;=\; r_A(\beta)\,, \tag{2.3}$$

for any $\beta \in \mathbb{C}^d$ (cf. [**17**]).

We may assume that all $b_j \neq 0$ or, equivalently, that the set A is minimally dependent, i.e. a *circuit*. Indeed, suppose $b_s = 0$, then we can find an integral $d \times s$ matrix, of rank d

$$A_1 = \begin{pmatrix} \tilde{A} & 0 \\ 0 & 1 \end{pmatrix} \tag{2.4}$$

such that $A_1 \cdot b = 0$. Now, $f(x_1, \dots , x_s)$ is A_1-hypergeometric with parameters $\gamma = (\gamma_1, \dots , \gamma_d)$ if and only if

$$f(x_1, \dots , x_s) \;=\; x_s^{\gamma_d} \cdot \tilde{f}(x_1, \dots , x_{s-1})$$

where $\tilde{f}$ is $\tilde{A}$-hypergeometric with parameters $\tilde{\gamma} = (\gamma_1, \dots , \gamma_{d-1})$. Thus, the study of rational A-hypergeometric functions reduces to that of rational functions which are $\tilde{A}$-hypergeometric.

Suppose then that A is a circuit and assume that

$$b_j > 0 \text{ for } j = 1, \dots , m\,; \qquad b_j < 0 \text{ for } j = m + 1, \dots , s. \tag{2.5}$$

We note that the higher-order operators (1.10) are generated by the single operator

$$D_A := \partial^{b_+} - \partial^{b_-} \tag{2.6}$$

where $b_+ = (b_1, \ldots, b_m, 0, \ldots, 0)^T$ and $b_- = (0, \ldots, 0, -b_{m+1}, \ldots, -b_s)^T$.

Given any A-homogeneous function φ with homogeneity $A.v$, there exists a one variable function f such that

$$\phi(x_1, \ldots, x_{d+1}) = x^v f(\prod_i (x_i/b_i)^{b_i}) = x^v f((x/b)^b). \tag{2.7}$$

PROPOSITION 2.1. *Set P_v the one variable operator (in a variable z):*

$$P_v := \prod_{i=1}^{m} \prod_{j=o}^{b_i-1} (\Theta + \frac{v_i - j}{b_i}) - z \prod_{i=m+1}^{d+1} \prod_{j=o}^{|b_i|-1} (\Theta + \frac{v_i - j}{b_i}),$$

where we denote $\Theta := zd/dz$. Then it holds that $D_A(\phi) = 0$ iff $P_v(f) = 0$.

PROOF. First note that $D_A(\phi) = 0$ iff $\tilde{D}_A := \left(\prod_{i=1}^m (x_i)^{b_i}\right) D_A$ annihilates ϕ. Set $\Theta_i := x_i \partial_i$. Using the identities $\Theta_i x^v = x_v(\Theta_i + v_i)$ and $\Theta_i f((x/b)^b) = b_i \Theta f((x/b)^b)$ it follows that

$$\tilde{D}_A = \prod_{i=1}^{m} \prod_{j=0}^{b_i-1} (\Theta_i - j) - \left(\prod_i (x_i)^{b_i}\right) \prod_{i=m+1}^{d+1} \prod_{j=0}^{|b_i|-1} (\Theta_i - j) \tag{2.8}$$

and

$$\tilde{D}_A(\phi) = x^v \left(\prod_{i=1}^{m} b_i^{b_i}\right) P_v(f)((x/b)^b).$$

■

REMARK 2.2. Note that we may always change any integer vector v by a vector v' such that $Av = Av'$ and $0 \leq v'_1 \leq b_1 - 1$. For such v', $P_{v'}$ has has the form (1.8). In any case, the only singular points of P_v are $0, 1$ and ∞, and all three are regular singular points. Proposition 2.1 translates between A-hypergeometric functions for codimension one configurations A and generalized univariate hypergeometric functions attached to special values of the parameters (from which one can read the generator b of the lattice $\mathcal{L}_A$).

EXAMPLE 2.3. Consider the codimension one system associated with the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}. \tag{2.9}$$

Given $\beta \in \mathbb{Z}^2$, let $v = (0, v_1, v_2) \in \mathbb{Z}^3$ such that $A \cdot v = \beta$. In this case,

$$P_v = \Theta(\Theta + v_2) - z(\Theta - \frac{v_1}{2})(\Theta - \frac{v_1}{2} + \frac{1}{2}). \tag{2.10}$$

The kernel $\mathcal{L}_A$ has rank 1 and is generated by $(1, -2, 1)$. The fact that the generator has a coordinate equal to 2 is reflected by the two parameters differing by 1/2 in the operator P_v.

Translating via Proposition 2.1 the results in [**10**], we know for example that when $v_1 > 0$ and $v_1 + 2v_2 < 0$, logarithmic solutions arise.

Let $\beta \in \mathbb{Z}^d$ and $c \in \mathbb{Z}^s$ such that $A \cdot v = \beta$. A rational non Laurent polynomial A-hypergeometric function $F(x)$ with homogeneity β has two different series expansions F_+, F_- at each of the vertices of the discriminant

$$\Delta_A := b_+^{b_+} x^{b_-} - (-1)^p b_-^{b_-} x^{b_+} = b_+^{b_+} x^{b_-} (1 - (x/b)^b)$$

(cf. ([**20**]), which will converge for $|(x/b)^b|$ greater than 1 and less than 1 (may be different also from 0) respectively. If we write, as in (2.7),

$$F(x_1, \dots, x_{d+1}) = x^v f((x/b)^b), \tag{2.11}$$

this corresponds to the fact that 1 is a pole of f (and may be also 0 or ∞). Taking into account that derivatives preserve "negative supports" of monomials, may be after susbstracting an A-hypergeometric Laurent polynomial, the Laurent series

$$F_-(x) = x^v \sum_{n=0}^{\infty} c_n \left((x/b)^b\right)^n, \; c_n \in \mathbb{C},$$

is A-hypergeometric and rational (cf. [**30**, §3.4]), and we may suppose that

$$v_j < 0 \Leftrightarrow j > m. \tag{2.12}$$

]From the recursion relations for the coefficients derived from the operator (2.8), we deduce that up to constant

$$F_-(x) = x^v \sum_{n=0}^{\infty} \frac{\prod_{j>m}(-v_j - nb_j - 1)!}{\prod_{j\le m}(v_j + nb_j)!} \left((-1)^p \frac{x^{b_+}}{x^{b_-}}\right)^n. \tag{2.13}$$

Symmetrically, there is another A-hypergeometric rational Laurent series expansion F_+ in negative powers of $(x/b)^b$.

Example 2.4. Consider the Gauss configuration in Example 1.1 and $\beta = (-2, -1, -2)$. In this case, the rational function

$$F(x) \; := \; \frac{x_1}{x_4\,(x_3x_4 - x_1x_2)}$$

is A-hypergeometric. The Laurent series expansion of F in powers of $(x_1x_2)/(x_3x_4)$ is equal to the series F_-, while the expansion of F in powers of $(x_3x_4)/(x_1x_2)$ equals the sum of $-F_+$ and the A-hypergeometric Laurent polynomial $-1/(x_2x_4)$.

Example 2.5. Consider again the configuration (2.9). In [**10**, Theorem 1.10] we prove that the only rational A-hypergeometric functions are Laurent polynomials, and we propose explicit bases of solutions for all integer homogeneities, which can be translated into the solution to the corresponding equations $P_v(y) = 0$, where P_v are given by (2.10). The Laurent series F_- and F_+ defined by (2.13) are hypergeometric with non-trivial domains of convergence; in fact $F_-(x) = (x_2^2 - 4x_1x_3)^{-1/2}$ and

$$F_+(x) = 4\,\arcsin\left(\left(\frac{{x_2}^2}{4x_1x_3}\right)^{1/2}\right) \frac{1}{\sqrt{4x_1x_3 - {x_4}^2}}.$$

Note that while $F_-(x)$ is algebraic, $F_+(x)$ is not.

Definition 2.6. We will say that a circuit A is *balanced* if it satisfies $d+1 = 2m$ (i.e. the number of positive and negative coordinates in a genererator b of the lattice $\mathcal{L}_A$ coincide) and, after reordering if necessary, $b_i = -b_{m+i}$, $i = 1, \dots, m$. Otherwise, we will call A *unbalanced*

Remark 2.7. Balanced circuits are very special. They clearly satisfy the regularity condition (1.9). Moreover, it is easy to see that up to multiplication by an element of $GL(s, \mathbb{Q})$ the matrix A of a balanced circuit is of the form

$$A = \begin{pmatrix} I_m & I_m \\ 0 & \tilde{A} \end{pmatrix}$$

where $\tilde{A}$ is a $(m-1)\times m$ integral matrix of maximal rank such that $\tilde{A}\cdot b_+ = 0$, $b_+ = (b_1,\dots,b_m)$.

For $d=3$ for example, a circuit consists of 4 points in a two dimensional plane, such that no 3 of them are collinear, and such that the cuadrilateral with these vertices has a pair of parallel sides. Note that for d even there are no balanced circuits. In particular, 3 collinear points form an unbalanced circuit.

The following Theorem characterizes those circuits for which there exists a rational hypergeometric function which is not a Laurent polynomial. When A is a balanced circuit, it is easy to see that $\Delta(A)^{-1}$ is a rational A-hypergeometric function (cf. [**13**], Lemma 2.2). In fact, only balanced circuits admit rational A-hypergeometric functions with infinite Laurent series expansions.

Theorem 2.8. *Let A be a circuit in $\mathbb{R}^d$. Then, there exists a rational A-hypergeometric function, which is not a Laurent polynomial, if and only if A is balanced.*

Corollary 2.9. *Let $A\in\mathbb{Z}^{d\times s}$ be a codimension-one configuration. There exists a rational A-hypergeometric function which is not a Laurent polynomial if and only if there exists $M\in GL(d;\mathbb{Q})$ such that*

$$M\cdot A \;=\; \begin{pmatrix} \tilde{A} & 0 \\ 0 & I_k \end{pmatrix},$$

where I_k is a $k\times k$ identity matrix and $\tilde{A}$ is a balanced circuit in $\mathbb{Z}^{d-k}$.

Proof. The only if part follows from Theorem 2.8 and the discussion at the beginning of this section. To prove the converse, suppose $\tilde{A}$ is balanced circuit in $\mathbb{Z}^{d-k}$. It follows from Theorem 2.8 that there exists a rational $\tilde{A}$ hypergeometric function $\tilde{f}(x_1,\dots,x_{s-k})$ with homogeneity $\tilde{\beta}=(\beta_1,\dots,\beta_{d-k})\in\mathbb{Z}^d$. Then,

$$f(x_1,\dots,x_s) \;:=\; \tilde{f}(x_1,\dots,x_{s-k})$$

is A-hypergeometric of exponent $M^{-1}\cdot(\beta_1,\dots,\beta_{d-k},0,\dots,0)\in\mathbb{Z}^d$. ∎

A proof of Theorem 2.8 can be found in [**13**], Th.2.3. Instead, we stress now the relation with classical univariate hypergeometric functions and give, with a similar proof, a complete characterization of those hypergeometric Laurent series which are the Laurent expansion of a hypergeometric rational function, and thus of those homogeneities β for which there exists a rational A-hypergeometric function with homogeneity β.

We first need the following slight generalization of [**28**, Lemma 2.1], whose proof is left to the reader.

Lemma 2.10. *Let $\rho(z)$ be a rational function. Let $\Sigma=\{\sigma_1,\dots,\sigma_p\}$ and $\Sigma'=\{\sigma'_1,\dots,\sigma'_q\}$ denote the poles and zeroes of ρ listed with multiplicity. There exists a polynomial $g(z)$ such that $\rho(z)=g(z+1)/g(z)$ if and only if $p=q$ and it is possible to reorder the collections Σ and Σ' in such a way that $\sigma_j - \sigma'_j\in\mathbb{N}$.*

Because of Theorem 2.8, we may restrict ourselves to the case when A is a balanced circuit, and moreover, $b_{m+i}=-b_i$, for all $i=1,\dots,m$, $\beta\in\mathbb{Z}^d$, $A\cdot v=\beta$ and (2.12) holds.

Define

$$\mu'_i \;:=\; -(v_i+1)/b_i\,,\quad i=1,\dots,m\,.$$

and let

$$\Sigma'_v \; := \; \{\, \mu_i - j/b_i \,;\; j = 0, \dots , b_i - 1 \,;\; j = 1, \dots , m\}.$$

Define also

$$\mu_i \; := \; v_{m+i}/b_i \,, \quad i = 1, \dots , m \,.$$

and let similarly

$$\Sigma_v \; := \; \{\, \mu_i - j/b_i \,;\; j = 0, \dots , b_i - 1 \,;\; i = 1, \dots , m\}.$$

Note that both Σ and Σ' have cardinality p. We list for simplicity

$$\Sigma_v = \{\sigma_1, \dots , \sigma_p\} \;; \quad \Sigma'_v = \{\sigma'_1, \dots , \sigma'_p\} \,.$$

Theorem 2.11. *The series $F_-(x)$ in (2.13) is the Laurent expansion of a rational function if and only if it is possible to order the sets Σ_v and Σ'_v in such a way that for every $j = 1 \dots , p$, $\sigma_j \; - \; \sigma'_j \in \mathbb{N}$.*

Proof. The Laurent series $F_-(x)$ is the expansion of a rational function if and only if the associated univariate power series $f_-(z)$ defined as in (2.7) by $F_-(x) = x^v f_-((x/b)^b)$ is a rational function of z. Since A is a balanced circuit, $b^b = (-1)^p$. We then have

$$f_-(z) \; := \; \sum_{n=0}^{\infty} \frac{\prod_{i=1}^m (b_i n - v_{m+i} - 1))!}{\prod_{i=1}^m (b_i n + v_i))!} \, z^n \tag{2.14}$$

Since $f_-(z)$ has no pole at the origin, it may be written as a quotient

$$f_-(z) \; = \; \frac{P(z)}{(1-z)^{r+1}}$$

with $P \in \mathbb{C}[z]$. Note that thanks to the fact that A is balanced, the coefficients $\gamma(n)$ of the series (2.14) are rational. It follows from [**32**, Corollary 4.3.1] that $f_-(z)$ is rational if and only if $\gamma(n)$ is polynomial in n. The function

$$\rho(z) := \frac{\gamma(z+1)}{\gamma(z)}$$

is also rational. Moreover, Σ_v and Σ_v describe the poles and zeroes of ρ. When γ is polynomial, we are done by Lemma 2.10. With respect to the if part, we know by the same Lemma that there exists a polynomial g such that $g(z+1)/g(z) = \rho(z) = \gamma(z+1)/\gamma(z)$. Then, as $\gamma(n)$ is rational in n, the function $r := \gamma/g$ is also rational and verifies $r(z+1) = r(z)$ for all z, and so is constant, from which we deduce that γ is a polynomial. ■

Corollary 2.12. *Suppose that the series $F_-(x)$ is the Laurent expansion of a rational function. i.e. that Σ and Σ' may be ordered in such a way that for every $j = 1 \dots , p$, $\sigma'_j \; - \; \sigma_j \in \mathbb{N}$. Then, $F_-(x) = Q(x)/\Delta_A^{r+1}$, where $r := \sum_{i>m}(-v_i) - \sum_{i \le m} v_i - m$ and $Q \in \mathbb{Q}[x]$ is a polynomial of degree at most r.*

We now relate A-hypergeometric functions associated with a circuit A and classical generalized hypergeometric functions. Recall that for $\mu \in \mathbb{C}$ and $n \in \mathbb{N}$, $(\mu)_n$ denotes the product

$$(\mu)_n = \mu \cdot (\mu+1) \dots \cdot (\mu+n-1)$$

Note that $(\mu)_n = 0$ for some $n \in \mathbb{N}$ if and only if $\mu \in \mathbb{Z}_{\le 0}$.

Let $(\mu; \mu') = (\mu_1, \dots , \mu_p; \mu'_1, \dots , \mu'_{p-1}) \in \mathbf{C}^{2p-1}$ where no μ'_i is a negative integer (or such that there exists an injective function $j(i)$ defined for all i for which μ'_i is a negative integer, such that $\mu_{j(i)}$ is a bigger negative integer). The classical hypergeometric function

$$ {}_pF_{p-1}((\mu;\mu'),z) := \sum_{n=0}^{\infty} \frac{(\mu_1)_n \dots (\mu_p)_n}{(\mu'_1)_n \dots (\mu'_{p-1})_n} \frac{z^n}{n!} \tag{2.15} $$

is then defined and, clearly, it is a polynomial if and only if there exists an index i such that $\mu_i \in \mathbb{Z}_{\leq 0}$. Since ${}_pF_{p-1}((\mu;\mu'),z)$ is annihilated by the (regular) operator $\Theta \prod_{i=1}^{p-1}(\Theta+\mu'_i-1) - z\ \prod_{i=1}^{p}(\Theta+\mu_i)$, its only possible poles are $0, 1$ and ∞. Since, by definition, it is holomorphic at the origin, it might be written as $P(z)/(1-z)^r$ for some $r \in \mathbb{N}$ and $P \in \mathbb{C}[z]$.

Translating into this language the proof of Theorem 2.11 we identify now all other rational classical hypergeometric functions:

THEOREM 2.13. *Let $(\mu;\mu') = (\mu_1,\dots,\mu_p;\mu'_1,\dots,\mu'_{p-1}) \in \mathbb{C}\setminus\mathbb{Z}^{2p-1}_{\leq 0}$ The classical hypergeometric function ${}_pF_{p-1}((\mu;\mu'),z)$ is rational if and only if up to reordering, $\mu_i - \mu'_i \in \mathbb{Z}\geq 0$, $i = 1,\dots,p-1$, and $\mu_p \in \mathbb{Z}_{\geq 1}$. Moreover, in this case, there exists a polynomial P such that ${}_pF_{p-1}((\mu;\mu'),z) = P(z)/(1-z)^r$, where $r := \sum_{i=1}^{p}\mu_i - \sum_{i=1}^{p-1}\mu'_i$.*

REMARK 2.14. The hypothesis $\mu_p \in \mathbb{Z}_{\geq 1}$ reflects the fact that there's a p-th parameter equal to 1 in the denominator producing the factor $n! = (1)_n$.

3. Cayley configurations, residues, and rational A-hypergeometric functions

Theorem 2.8 is combined in [**13**] with a careful analysis of restrictions of A-discriminants to subconfigurations, to find severe restrictions on a configuration A in order to admit rational non Laurent polynomial A-hypergeometric functions. For instance, for any configuration A such that each facial subset of A has a relative interior point, all rational solutions are Laurent polynomials . So, there are "very few" configurations A having rational A-hypergeometric functions with infinite series expansions. However, such functions are very interesting!

There is a general procedure to build rational A-hypergeometric functions based on the notion of toric residue ([**14**],[**9**], [**8**],[**13**]), which corresponds to the total sum of local Grothendieck residues ([**22**], Ch.5; [**38**]). There are analytic, geometric and algebraic definitions of local multidimensional residues associated with a family of n-variate polynomials $f_1,\dots,f_n$ with an isolated zero at a point ξ, all of them are natural extensions of the following definition in case ξ is a simple zero: given any other regular function g at ξ, the local residue $\mathrm{Res}_{f,\xi}(g)$ of g with respect to $f_1,\dots,f_n$ equals

$$ \mathrm{Res}_{f,\xi}(g) = \frac{g(\xi)}{J_f(\xi)}, \tag{3.1} $$

where J_f denotes the Jacobian determinant of $f_1,\dots,f_n$. In fact, a residue is attached to a differential form, but in the affine case (resp. in the case of the torus $(\mathbb{C}^*)^n$), the standard n-form $dz_1\wedge\dots\wedge dz_n$ (resp. $dz_1/z_1\wedge\dots\wedge dz_n/z_n$) allows us to assign a residue to a function. With this terminology, (1.14) is written as $\mathrm{Res}_{f^a,\rho(x)}(t^{b-1})$.

One important property of the residue is that $\mathrm{Res}_{f,\xi}(J_f)$ is an integer number, namely, the (algebraic or geometric) i.e. traces are instances of residues, and in particular, we recover the coordinates of the common root. The main property of the residue operators is that they provide a natural local duality; namely, given g, $\mathrm{Res}_{f,\xi}(g \cdot h) = 0$ for any h if and only if g lies in the local ideal $I_{f,\xi}$ generated by $f_1, \ldots, f_n$. There are manifold versions of this statement (local and global in the algebraic and analytic settings), including "effective" versions useful for computer algebra (cf. for instance [**7**], [**12**]).

Fix $n+1$ configurations of integer points $A_0, \ldots, A_n$ in $\mathbb{Z}^n$ and let $f_0, \ldots, f_n$ be generic n-variate polynomials with these exponents

$$f_i(\mathbf{x}; t_1, \ldots, t_n) = \sum_{\alpha \in A_i} x_{\alpha,i} t^\alpha.$$

Then, for generic values of the parameters $\mathbf{x}$, $f_1, \ldots, f_n$ will have a finite set V of common zeroes in the torus and f_0 will not vanish on V.

Definition 3.1. The Cayley configuration A_C associated with $A_0, \ldots, A_n$ is the configuration in $\mathbb{Z}^{2n+1}$ defined by

$$A_C \;=\; \{e_0\} \times A_0 \,\cup\, \{e_1\} \times A_1 \,\cup\, \cdots \,\cup\, \{e_n\} \times A_n \tag{3.2}$$

Remark 3.2. Note that by Remark 2.7, balanced circuits are (after reordering) Cayley configurations.

Define also

Definition 3.3. The Cayley configuration $\tilde{A}_C$ associated with $A_1, \ldots, A_n$ is the configuration in $\mathbb{Z}^{2n}$ defined by

$$\tilde{A}_C \;=\; \{e_1\} \times A_1 \,\cup\, \cdots \,\cup\, \{e_n\} \times A_n \tag{3.3}$$

We then have

Proposition 3.4. *For any $m \in \mathbb{Z}^n$,*

- *the local residue* $\mathrm{Res}_{f,\xi}(t^m)$ *of the Laurent monomial t^m at any point $\xi \in V$ is $\tilde{A}_C$-hypergeometric function of the coefficients of $f_1, \ldots, f_n$ with homogeneity $(-1, \ldots, -1, -m_1 - 1, \ldots, -m_n - 1) \in \mathbb{Z}^{2n}$.*
- *the local residue* $\mathrm{Res}_{f,\xi}(t^m/f_0)$ *of the rational function t^m/f_0 at any point $\xi \in V$ is A_C-hypergeometric function of the $\mathbf{x}$ variables with homogeneity $(-1, \ldots, -1, -m_1 - 1, \ldots, -m_n - 1) \in \mathbb{Z}^{2n+1}$.*

Residues can be given an integral representation and Proposition 3.4 may be proved by differentiating under the integral sign. One can also give a complete algebraic proof, following [**2**], based on the equality (3.1).

Given any monomial t^m, $m \in \mathbb{Z}^n$, we can also define the global residue

$$\mathrm{Res}_{f_1, \ldots, f_n}(t^m/f_0)$$

as the sum over all common roots

$$\mathrm{Res}_{f_1, \ldots, f_n}(t^m/f_0) = \sum_{\xi \in V} \mathrm{Res}_{f,\xi}(t^m/f_0). \tag{3.4}$$

Global residues are in fact cohomological objects in a suitable compact toric variety containing $(\mathbb{C}^*)^n$ as a dense open set (cf. [**9**]). Since the A-hypergeometric system is linear, the global residue is also an A_C-hypergeometric function with the same homogeneity, and moreover it is a rational function of $\mathbf{x}$.

It has been observed in [**36**], the coordinates of the common roots of $f_1, \ldots, f_n$ are not in general $\tilde{A}_C$-hypergeometric, even in the case of linear polynomials. In the case of monomial curves (1.2) it is shown in [**10**] that for (integer) homogeneities with negative first coordinates, all rational A-hypergeometric functions are constant multiples of global residues. In the multivariate setting, interesting combinatorial problems arise and we expect that global residues become a key ingredient for the description of all rational A_C-hypergeometric functions. We end with a description of the denominator of the global residues for special values of m.

Suppose that $A_0, \ldots, A_n$ is an essential configuration. This means that the Minkowski sum $\sum_{i\in I} A_i$ has affine dimension at least $|I|$ for every proper subset I of $\{0, \ldots, n\}$. Then, the sparse resultant $R_{A_0,\ldots,A_n}$ is a non constant (and non monomial!) polynomial in the variables $\mathbf{x}$ (cf. [**34**]) which vanishes for all those values of the parameters such that $f_0, \ldots, f_n$ have a common root in the torus. In [**11**] we described the denominator of the global residues $\sum_{\xi\in V} \mathrm{Res}_{f,\xi}(t^m)$ of Laurent monomials in terms of explicit powers of resultants attached to the facets of the convex hulls of $A_1, \ldots, A_n$. From the description of the singular locus of the A_C-hypergeometric system, we know that the denominator of $\mathrm{Res}_{f_1,\ldots,f_n}(t^m/f_0)$ is a product of powers of facet discriminants associated with the Cayley configuration A_C. In fact, in case $A_0, \ldots, A_n$ is essential it happens that the discriminant $\Delta(A_C)$ coincides with the sparse resultant $R_{A_0,\ldots,A_n}$ (cf. [**13**, Prop. 5.1]; [**20**]). Mimicking the proof in [**11**, Th. 1.4], we can moreover prove

Proposition 3.5. *For any $m \in \mathbb{Z}^n$ which lies in the interior of the Minkowski sum of the convex hulls $P_0, \ldots, P_n$ of $A_0, \ldots, A_n$, there exists a polynomial $P \in \mathbb{Q}[\mathbf{x}]$ such that*

$$\mathrm{Res}_{f_1,\ldots,f_n}\Big(\frac{t^{m+\mathbf{1}}}{f_0}\Big) = \frac{P_m}{R_{A_0,\ldots,A_n}},$$

where $\mathbf{1}$ denotes the vector with all coordinates equal to 1.

Proof. As noted above, for values of $\mathbf{x}$ in a Zariski open set, $f_0, \ldots, f_n$ have no common zeroes and V is finite and contained in the torus. Moreover, we can suppose that all common zeroes of $f_1, \ldots, f_n$ are simple and, therefore, for any $\xi \in V$,

$$\mathrm{Res}_{f,\xi}\Big(\frac{t^{m+\mathbf{1}}}{f_0}\Big) = \frac{t^{m+\mathbf{1}}(\xi)}{f_0(\xi)\,\cdot\, J_f(\xi)} \;=\; \frac{g_\xi(x_1, \ldots, x_n)}{f_0(\xi)\,\cdot\, h_\xi(x_1, \ldots, x_n)}\,, \tag{3.5}$$

where the symbol x_i stands for the vector $(x_{\alpha,i} : \alpha \in A_i)$ of coefficients of f_i, and g_ξ, h_ξ are algebraic functions in these coefficients.

We now sum (3.5) over all points ξ in V. For any choice of monomials $t^{a_i}, i = 0, \ldots, n$, the residue of $t^{m+\mathbf{1}}/f_0$ with respect to $f_1, \ldots, f_n$ equals the residue of $t^{m+\mathbf{1}+\sum_i a_i}/t^{a_0} f_0$ with respect to $t^{a_1} f_1, \ldots, t^{a_n} f_n$, and we may thus suppose that each A_i contains the origin. Then, the affine lattice L generated by $A_0 + \ldots A_n$ agrees with the linear lattice it generates. We assume by simplicity that $L = \mathbb{Z}^n$. If this is not the case, we can argue as in the proof of Theorem 1.4 in [**11**]. Therefore,

$$\mathrm{Res}_{f_1,\ldots,f_n}\Big(\frac{t^{m+\mathbf{1}}}{f_0}\Big) = \sum_{\xi\in V} \frac{1}{f_0(\xi)}\, \frac{a_\xi(x_1, \ldots, x_n)}{b_\xi(x_1, \ldots, x_n)}\,.$$

This expression depends rationally on $x_0, x_1, \ldots, x_n$. This implies that there exist polynomials $G(x_0, \ldots, x_n)$ and $H(x_1, \ldots, x_n)$ such that

$$\mathrm{Res}_{f_1,\ldots,f_n}(\frac{t^{m+\mathbf{1}}}{f_0}) = \frac{G(x_0, x_1, \ldots, x_n)}{(\prod_{\xi \in V} f_0(\xi)) \cdot H(x_1, \ldots, x_n)}.$$

It follows from [**27**, Theorem 1.1] that

$$\prod_{\xi \in V} f_0(\xi) \quad = \quad R_{A_0,\ldots,A_n}(x_0, x_1, \ldots, x_n) \cdot C(x_1, \ldots, x_n)$$

for some rational function C. Therefore, there exist polynomials G_0, H_0 such that

$$\mathrm{Res}_{f_1,\ldots,f_n}(\frac{t^{m+\mathbf{1}}}{f_0}) = \frac{G_0(x_0, u_1, \ldots, x_n)}{R_{A_0,\ldots,A_n}(x_0, x_1, \ldots, x_n) \cdot H_0(x_1, \ldots, x_n)}.$$

Now the key point is that because of the hypothesis that m lies in the interior of $P = P_0 + \ldots + P_n$, and passing to a toric residue in the toric variety X_P associated with P, we deduce from [**9**, Th. 0.4] (see also [**11**, Prop. 1.3]) that for any $i = 1, \ldots, n$, the sum of the local residues of $\frac{t^{m+\mathbf{1}}}{f_i}$ with respect to $f_0, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n$ at each of the points of intersection of these n polynomials, coincides up to sign with $\mathrm{Res}_{f_1,\ldots,f_n}(\frac{t^{m+\mathbf{1}}}{f_0})$. We can thus repeat the previous argument for any other index $i = 1, \ldots, n$ playing the role of the index 0, and the Proposition follows. ■

Remark 3.6. For any $m \in \mathbb{Z}^n$, we can similarly prove that the denominator of the global residue of t^m/f_0 with respect to $f_1, \ldots, f_n$ equals the resultant $R_{A_0,\ldots,A_n}$ times explicit powers of the facet resultants associated with $f_1, \ldots, f_n$.

Acknowledgements: I wish to thank S. Ofman and F. Norguet for inviting me to their enjoyable Colloque. I am also grateful to E. Cattani and B. Sturfmels for their friendship and support. This expository account grew out of our joint work on rational hypergeometric functions.

References

[1] A. Adolphson, *Hypergeometric functions and rings generated by monomials.* Duke Math. J. **73** (1994), 269–290.

[2] Adolphson, Alan and Sperber, Steven, *Differential modules defined by systems of equations.* Rend. Sem. Mat. Univ. Padova 95 (1996), 37–57.

[3] K. Aomoto, *Hypergeometric functions—their past, present and ....* Sūgaku **45** (1993), no.3, 208–220.

[4] V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties.* J. Algebraic Geometry **3** (1994), 493–535.

[5] F. Beukers and G. Heckmann. *Monodromy for the hypergeometric function* ${}_nF_{n-1}$. Invent. math. **95** (1989), 325–354.

[6] R. Birkeland, *Über die Auflösung algebraischer Gleichungen durch hypergeometrische Funktionen.* Mathematische Zeitschrift **26** (1927), 565–578.

[7] C. Berenstein and A. Yger, *Residue calculus and effective Nullstellensatz* . Amer. J. Math. 121 (1999), no. 4, 723–796.

[8] E. Cattani and A. Dickenstein, *A Global View of Residues in the Torus.* Journal of Pure and Applied Algebra **117 & 118** (1997), 119–144.

[9] E. Cattani, D. Cox and A. Dickenstein. *Residues in toric varieties.* Compositio Math. 108 (1997), no. 1, 35–76.

[10] E. Cattani, C. D'Andrea, and A. Dickenstein, *Rational solutions of the A-hypergeometric system associated with a monomial curve.* Duke Math. J. **99** (1999), 179–207.

[11] E. Cattani, A. Dickenstein, and B. Sturmfels, *Residues and resultants.* J. Math. Sci. Univ. Tokyo **5** (1998), 119–148.

[12] E. Cattani, A. Dickenstein, and B. Sturmfels, *Computing Multidimensional Residues*. Algorithms in algebraic geometry and applications (Santander, 1994), 135–164, Progr. Math., 143, Birkhuser, Basel, 1996.

[13] E. Cattani, A. Dickenstein, and B. Sturmfels, *Rational Hypergeometric Functions*. Preprint, 1999, http://xxx.lanl.gov/abs/math/9911030.

[14] D. Cox, *Toric Residues* . Arkiv för Matematik **34** (1996), 73–96.

[15] C.F. Gauss, *Disquisitiones generales circa seriem infinitam*

$$1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\ 1\ 2}x^2 + \dots.$$

Comment. Soc. Regiae Sci. Götingen (1812).

[16] I. M. Gel'fand, M. I. Graev, and A. Zelevinsky, *Holonomic systems of equations and series of hypergeometric type*. Dokl. Akad. Nauk SSSR **295** (1987), 14–19.

[17] I. M. Gel'fand, A. Zelevinsky, and M. Kapranov, *Hypergeometric functions and toral manifolds*. Functional Analysis and its Applications **23** (1989), 94–106.

[18] I. M. Gel'fand, M. Kapranov, and A. Zelevinsky, *Generalized Euler integrals and A-hypergeometric functions*. Advances in Mathematics **84** (1990), 255–271.

[19] I. M. Gel'fand, A. Zelevinsky, and M. Kapranov, *A-discriminants and Cayley-Koszul complexes*. Soviet Math. Doklady **40** (1990), 239–243.

[20] I. M. Gel'fand, M. Kapranov, and A. Zelevinsky, Discriminants, Resultants, and Multidimensional Determinants, Birkhäuser, Boston, 1994.

[21] I. M. Gel'fand, A. Zelevinsky, and M. Kapranov, *Correction to: "Hypergeometric functions and toral manifolds."* Functional Analysis and its Applications **27** (1994), 295.

[22] P. Griffiths and J. Harris, Principles of Algebraic Geometry. John Wiley & Sons, New York, 1978.

[23] S. Hosono, B.H. Lian, and S.-T. Yau, *Maximal degeneracy points of GKZ systems*. Journal of the American Mathematical Society **10** (1997), 427–443.

[24] K. Mayr, *Über die Auflösung algebraischer Gleichungssysteme durch hypergeometrische Funktionen*. Monatshefte für Mathematik und Physik **45** (1937), 280–313.

[25] F. Norguet, *Résoution des systèmes de deux équations algébriques á l'aide de fonctions hypergéométriques*. C.R.Acad.Sci.Paris Sér. I t. 254 (1962), 608–610.

[26] F. Norguet, *Formules explicites pour la résoution des systèmes de deux équations algébriques á l'aide de fonctions hypergéométriques*. C.R.Acad.Sci. Paris Sér.I t. 254 (1962), 801–802.

[27] P. Pedersen and B. Sturmfels, *Product formulas for resultants and Chow forms*. Mathematische Zeitschrift **214** (1993) 377–396.

[28] M. van der Put and M. F. Singer, Galois theory of difference equations, Lecture Notes in Mathematics **1666**. Springer-Verlag, Berlin Heidelberg, 1997.

[29] B. Riemann, *Beiträge zur Theorie der durch Gauss'sche Reihe $F(\alpha,\beta,\gamma,x)$ darstellbaren Functionen*. K. Gess. Wiss. Göttingen, **7**, 1–24 (1857)

[30] M. Saito, B. Sturmfels, and N. Takayama, Grö bner deformations of hypergeometric differential equations. Algorithms and Computation in Mathematics **6**, Springer-Verlag, Heidelberg, 1999.

[31] L. J. Slater, Generalized hypergeometric functions. Cambridge University Press, Cambridge, 1966.

[32] R. Stanley, Enumerative Combinatorics, Volume I, Cambridge Studies in Advanced Mathematics **49**. Cambridge University Press, Cambridge, 1997.

[33] J. Stienstra, *Resonant hypergeometric systems and mirror symmetry*. Integrable Systems and Algebraic Geometry (Proceedings of the Taniguchi Symposium 1997), 412–452. Edited by M.H. Saito, Y. Shimizu, and K. Ueno. World Scientific, Singapore.

[34] B. Sturmfels: *On the Newton polytope of the resultant*. Journal of Algebraic Combinatorics **3** (1994) 207–236.

[35] B. Sturmfels, Gröbner bases and convex polytopes. American Mathematical Society, University

Lectures Series, No. 8, Providence, Rhode Island, 1995.

[36] B. Sturmfels, *Solving Algebraic Equations in Terms of $\mathcal{A}$-hypergeometric series*. 8-th international Conference on Formal Power Series and Algebraic Combinatorics, Minneapolis, June 1996, to appear in Discrete Mathematics.

[37] N. Takayama, Kan: A system for computation in algebraic analysis, source code available at `http://www.math.s.kobe-u.ac.jp/KAN/` (1991)

[38] A. Tsikh, Multidimensional Residues and Their Applications. American Mathematical Society, Providence, 1992.

Departamento de Matemática, FCEyN, Universidad de Buenos Aires, Ciudad Universitaria, Pab. I, (1428) Buenos Aires, Argentina
E-mail address: alidick@dm.uba.ar