Multihomogeneous resultant matrices

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ABSTRACT
Multihomogeneous structure in algebraic systems is the first step away from the classical theory of homogeneous equations towards fully exploiting arbitrary supports. We propose constructive methods for resultant matrices in the entire spectrum of resultant formulae, ranging from pure Sylvester to pure Bézout types, including hybrid matrices. Our approach makes heavy use of the combinatorics of multihomogeneous systems, inspired by and generalizing certain joint results by Zelevinsky, and Sturmfels or Weyman [15, 18]. One contribution is to provide conditions and algorithmic tools so as to classify and construct the smallest possible determinantal formulae for multihomogeneous resultants. We also examine the smallest Sylvester-type matrices, generically of full rank, which yield a multiple of the resultant. The last contribution is to characterize the systems that admit a purely Bézout-type matrix and show a bijection of such matrices with the permutations of the variable groups. Interestingly, it is the same class of systems admitting an optimal Sylvester-type formula. We conclude with an example showing all kinds of matrices that may be encountered, and illustrations of our MAPLE implementation.

Keywords
Sparse resultant, multihomogeneous system, determinantal formula, Sylvester and Bézout type matrix, degree vector

1. INTRODUCTION
Resultants provide efficient ways for studying and solving polynomial systems by means of their matrices. This paper considers the sparse (or toric) resultant, which can exploit *a priori* knowledge on the support of the equations, and scales with it. We concentrate on unmixed (i.e. with identical supports) systems where the variables can be partitioned into groups so that every polynomial is homogeneous in each group. Such polynomials, and the resulting systems, are called *multihomogeneous*. Multihomogeneous structure is a first step away from the classical theory of homogeneous systems towards fully exploiting arbitrary sparse structure. Multihomogeneous systems are encountered in several areas including geometric modeling (eg. [2, 14, 19]), game theory and computational economics.

Known sparse resultant matrices are of any from a range of different types. On the one end of the spectrum are the *pure Sylvester-type* matrices, where the polynomial coefficients fill in the nonzero entries of the matrix; such is the coefficient matrix of linear systems, Sylvester's matrix for univariate polynomials, and Macaulay's matrix for homogeneous systems. On the other end are the *pure Bézout-type* matrices, i.e. matrices where the coefficients of the *Bezoutian* associated to the input polynomials fill in the nonzero entries of the matrix, whereas hybrid matrices, such as Dixon's, contain blocks of both pure types. The complete example in Sect. 6 shows the intricacy of such matrices. Hence the interest to describe them in advance in terms of combinatorial data, which allows for a structured matrix representation, based on quasi-Toeplitz or quasi-Hankel structure [9, 13].

Our work builds on [18] and their study of multihomogeneous systems through the determinant of a resultant complex. First, we give algorithmic methods for identifying and constructing *determinantal formulae* for the sparse resultant, i.e. matrices whose determinant equals the sparse resultant. The underlying resultant complex is made explicit and computational tools are derived in order to produce the smallest such formula. Second, we describe and construct the smallest possible pure Sylvester matrices, thus generalizing the results of [15] and [10, 13.2,Prop.2.2]. The corresponding systems include all systems for which exact Sylvester-type matrices are known. The third contribution of this paper is to offer sufficient and necessary conditions for systems to admit purely Bézout determinantal formulae, thus generalizing a result from [3]. It turns out that these are precisely the same systems admitting optimal Sylvester-type formulae! We also show a bijection of such matrices with the permutations on $\{1, \dots, r\}$, where r stands for the number of the variable groups. While constructing explicit Bézout-type formulae, we derive a precise description of the

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support of the Bezoutian polynomial.

This paper is organized as follows. The next section gives a formal background and Sect. 1.2 presents previous results and how they are generalized here. Sect. 2 provides some technical facts useful later. Sect. 3 offers bounds in searching for the smallest possible determinantal (hybrid) formulae. Sect. 4 and 5 characterize matrices of pure Sylvester and pure Bézout type respectively. In Sect. 6 we provide an explicit example of a hybrid resultant matrix for a multidegree for which neither pure Sylvester nor pure Bézout determinantal formulae exist; this example illustrates all possible morphisms that may be encountered with multihomogeneous systems. Our MAPLE implementation is described in Sect. 7. Due to length restrictions, only the most important or representative proofs are included; an Appendix contains most other proofs.

1.1 The Setting

Consider the r-fold product $X := \mathbb{P}^{l_1} \times \ldots \times \mathbb{P}^{l_r}$ of projective spaces of respective dimensions $l_1, \ldots, l_r$ over an algebraically closed field of characteristic zero, for some natural number r. We denote by $n = \sum_{k=1}^{r} l_k$ the dimension of X, i.e. the number of affine variables.

DEFINITION 1.1. *Consider $d = (d_1, \ldots, d_r) \in \mathbb{N}^r_{>0}$ and multihomogeneous polynomials $f_0, \ldots, f_n$ of degree d. The multihomogeneous resultant is an irreducible polynomial $R(f_0, \ldots, f_n) = R_{(l_1,\ldots,l_r),d}(f_0, \ldots, f_n)$ in the coefficients of $f_0, \ldots, f_n$ which vanishes iff the polynomials have a common root in X.*

This is an instance of the sparse resultant [10]. It may be chosen with integer coefficients, and it is uniquely defined up to sign by the requirement that it has relatively prime coefficients. The resultant polynomial is itself homogeneous in the coefficients of each f_i, with degree given by the multihomogeneous Bézout bound $\binom{n}{l_1,\ldots,l_r} d_1^{l_1} \cdots d_r^{l_r}$ [10, prop. 13.2.1]. This number is sometimes called the m-homogeneous bound [16].

Let V be the space of $(n+1)$ tuples $f = (f_0, \ldots, f_n)$ of multihomogeneous forms of degree d over X. Given a *degree vector* $m \in \mathbb{Z}^r$ there exists a finite complex $K_. = K_.(m)$ of free modules over the ring of polynomial functions on V [18], whose terms depend only on $(l_1, \ldots, l_r), d$ and m and whose differentials are polynomials on V satisfying:
(i) For every given f we can specialize the differentials in $K_.$ by evaluating at f to get a complex of finite-dimensional vector spaces.
(ii) This complex is exact iff $R(f_0, \ldots, f_n) \neq 0$.
We describe the terms in this complex in Sect. 2.

DEFINITION 1.2. *Given r and $(l_1, \ldots, l_r), (d_1, \ldots, d_r) \in \mathbb{N}^r$, define the* defect *vector $\delta \in \mathbb{Z}^r$ (just as in [15, 18]) by $\delta_k := l_k - \lceil \frac{l_k}{d_k} \rceil$. Clearly, this is a non-negative vector. We also define the* critical degree *vector $\rho \in \mathbb{N}^r$ by $\rho_k = (n+1)d_k - l_k - 1$*

LEMMA 1.3. *[18] For any $i \in [r] := \{1, \ldots, r\}$, $d_i l_i < d_i + l_i \Leftrightarrow \delta_i = 0 \Leftrightarrow \min\{l_i, d_i\} = 1$.*

1.2 Previous work

The complex with terms $K_\nu(m)$ described in the next section is known as the *Weyman complex.* For any choice of $l_1, \ldots, l_k$, d and m, the multihomogeneous resultant equals the determinant of the Weyman complex (for the corresponding monomial basis at each of the terms), which can be expressed as a quotient of products of subdeterminants extracted from the differentials in the complex. This way of defining the resultant was introduced by Cayley [10, appx A],[17], [18]. In the particular case in which the complex has just two terms, its determinant is nothing but the determinant of the only nonzero differential, which is therefore equal to the resultant. In this case, we say that there is a determinantal formula for the resultant and the corresponding degree vector m is called determinantal. In [18], the multihomogeneous systems for which a determinantal formula exists were classified; see also [10, sect. 13.2]. Their work, though, does not identify completely the corresponding morphisms, a question we partially undertake. We follow the results [4], which concerned the homogeneous case, inspired also by [12].

The main result of [15] was to prove that a determinantal formula of Sylvester type exists exactly when all defects are zero. In [15, Th.2] (recalled in [10, 13.2,Prop.2.2]) all such formulae are characterized by showing a bijection with the permutations of $\{1, \ldots, r\}$ and defined the corresponding degree vector m as in Def. 4.2 below. This includes all known Sylvester-type formulae, in particular, linear systems, systems of 2 univariate polynomials and bihomogeneous systems of 3 polynomials whose resultant is, respectively, the coefficient determinant, the Sylvester resultant and the Dixon resultant. In fact, Sturmfels and Zelevinsky characterized all determinantal Cayley-Koszul complexes, which are instances of the Weyman complexes when all the higher cohomologies vanish.

The incremental algorithm for sparse resultant matrices [7] relies on the determination of a degree vector m. When $\delta = 0$, it produces optimal Sylvester matrices by [15]. For other multihomogeneous systems, [7] heuristically produces small matrices, yet with no guarantee. For instance, on the system of Ex. 4.5, it finds a 1120×1200 matrix. The present paper explains the behavior of the algorithm, since the latter uses degree vectors following (4.2) defined by random permutations. Our results provide immediately the smallest possible matrix. More importantly, the same software constructs all Sylvester-type formulae described here.

Pure Bézout-type formulae were studied in [3] for unmixed systems whose support is the direct sum of what they call basis simplices, i.e. the convex hull of the origin and another l_k points, each lying on a coordinate axis. This includes the case of multihomogeneous systems. They showed that in this case a sparse resultant matrix can be constructed from the Bezoutian polynomial, to be defined in Sect. 5, though the corresponding matrix formula is not always determinantal. Their Cor. 4.2.1 states that for multihomogeneous systems with null defect vector, the Bézout formula becomes determinantal; Saxena had proved the special case of all $l_k = 1$ [14]. In [3, sect. 4.2] they indicate there are $r!$ such formulae and in Sect. 5 they study bivariate systems ($n = 2$) showing that then, these are the only determinantal formulae.

Sect. 5 proves these results in a different manner and characterizes the determinantal cases for multihomogeneous systems, showing that a null defect vector is a sufficient but also *necessary* condition for a determinantal formula for any n. We provide thus an affirmative answer, in the case of multihomogeneous systems, to the question on whether a generic polynomial system admits an optimal Sylvester-type formula iff it admits an optimal Bézout-type formula (cf. Def. 5.1) This had been proven for arbitrary systems only in the bivariate case [2]. In particular, we explicitly exhibit a choice of the differential in the Weyman complex in this case (cf. Thm. 5.13).

Studies exist [2, 4, 19] for dealing with hybrid formulae including Bézout-type blocks or pure Bézout matrices, and concentrate on the computation of such matrices. In particular, [19] elaborates on the relation of Sylvester and Bézout-type matrices (called Cayley-type there) and the transformations that link them. The theoretical setting together with Pfaffian formulae for resultants is addressed in [6].

2. PRELIMINARY OBSERVATIONS

Some facts from cohomology theory are necessary; see [11] for details. Given a *degree vector* $m \in \mathbb{Z}^r$, define, for $\nu \in \{-n, \dots, n+1\}$,

$$K_\nu(m) = \bigoplus_{p\in\{0,\dots,n+1\}} H^{p-\nu}(X, m - pd)^{\binom{n+1}{p}}, \qquad (1)$$

where for an integer r-tuple m', $H^q(X, m')$ denotes the q-th cohomology of X with coefficients in the sheaf $\mathcal{O}(m')$. The global sections $H^0(X, m')$ are identified with multihomogeneous polynomials of degree m'. By the Künneth formula, we have

$$H^q(X, m - pd) = \bigoplus_{\substack{j_k\in\{0,l_k\}\\ j_1+\cdots+j_r=q}} \bigotimes_{k=1}^{r} H^{j_k}(\mathbb{P}^{l_k}, m_k - pd_k),$$

where $q = p - \nu$ and the second sum runs over all integer sums $j_1 + \cdots + j_r = q, j_k \in \{0, l_k\}$. In particular, $H^0(\mathbb{P}^{l_k}, \alpha)$ is the space of all homogeneous polynomials in $l_k + 1$ variables with total degree α. We recall Bott's formulae for these cohomologies.

Proposition 2.1. *For any $m \in \mathbb{Z}^r$, $H^{l_k}(\mathbb{P}^{l_k}, m_k - pd_k) = 0 \Leftrightarrow m_k - pd_k \geq -l_k$, $H^0(\mathbb{P}^{l_k}, m_k - pd_k) = 0 \Leftrightarrow m_k - pd_k < 0$, for $k \in \{1, \dots, r\}$. Moreover,*

$$H^j(\mathbb{P}^{l_k}, m_k - pd_k) = 0, \forall j \neq 0, l_k,$$

$$\dim H^{l_k}(\mathbb{P}^{l_k}, m_k - pd_k) = \binom{-m_k + pd_k - 1}{l}_k,$$

$$\dim H^0(\mathbb{P}^{l_k}, m_k - pd_k) = \binom{m_k - pd_k + l_k}{l}_k,$$

$$\dim H^q(X, m-pd) = \sum_{\substack{j_k\in\{0,l_k\}\\ j_1+\cdots+j_r=q}} \prod_{k=1}^{r} \dim H^{j_k}(\mathbb{P}^{l_k}, m_k-pd_k),$$

$$\dim K_\nu(m) = \sum_{p\in[0,n+1]} \binom{n+1}{p} \dim H^{p-\nu}(X, m - pd).$$

We detail now the main results in [18]. They show (Lem. 3.3(a)) that a vector $m \in \mathbb{Z}^r$ is determinantal iff $K_{-1}(m) = K_2(m) = 0$. They also prove in Thm. 3.1 that a determinantal vector m exists iff $\delta_k \leq 2$ for all $k \in [r]$. To describe a differential in the complex from $K_\nu(m)$ to $K_{\nu+1}(m)$, one needs to describe all the morphisms $\delta_{p,p'}$ from the summand corresponding to an integer p to the summand corresponding to another integer p', where both $p, p' \in \{0, \dots, n+1\}$. [18, Prop. 2.5,2.6] proves this map is 0 when $p < p'$ and that, roughly speaking, it corresponds to a *Sylvester* map $(g_0, \dots, g_n) \to \sum_{i=0}^n g_i f_i$ when $p = p' + 1$, thus having all nonzero entries in the corresponding matrix given by coefficients of $f_0, \dots, f_n$. For $p > p' + 1$, the maps $\delta_{p,p'}$ are called higher-order differentials. By degree reasons, they cannot be given by Sylvester matrices. Thm. 2.10 also gives an explicit theoretical construction of the higher-order differentials in the pure Bézout case (cf. Def. 5.1).

3. DETERMINANTAL FORMULAE

This section addresses the computational problem of enumerating all determinantal vectors $m \in \mathbb{Z}^r$. The "procedure" of [18, Sect. 3] "is quite explicit but it seems that there is no nice way to parametrize these vectors", as stated in that paper. Instead, we bound the range of m to implement a computer search for them.

Given $k \in \{1, \dots, r\}$ and a vector $m \in \mathbb{Z}^r$ define as in [18]: $P_k(m) = \left\{p \in \mathbb{Z} : \frac{m_k}{d_k} < p \leq \frac{m_k+l_k}{d_k}\right\}$. Let $\tilde{P}_k(m)$ be the real interval $\left(\frac{m_k}{d_k}, \frac{m_k+l_k}{d_k}\right]$, so $P_k(m) = \tilde{P}_k(m) \cap \mathbb{Z}$. Using Lem. 3.3 in [18], it is easy to give bounds for all determinantal vector m for which all $P_k(m) \neq \emptyset$.

Lemma 3.1. *For a determinantal $m \in \mathbb{Z}^r$ and for all $k \in \{1, \dots, r\}$, $P_k \neq \emptyset$ implies,* $\max\{-d_k, -l_k\} \leq m_k \leq d_k(n+1) - 1 + \min\{d_k - l_k, 0\}$.

Now, $p \in P_k(m)$ iff $H^{l_k}(\mathbb{P}^{l_k}, m_k - pd_k) = H^0(\mathbb{P}^{l_k}, m_k - pd_k) = 0$. Thus a first guess could be that all determinantal vectors give $P_k(m) \neq \emptyset$. But this is not the case, as the example shows:

Example 3.2. Let $l = (1, 2), d = (2, 3)$. We focus on $m = (2\mu_1, 3\mu_2)$; both m below yield determinantal formulae M:

$$m = (4, 3) \Rightarrow \quad \tilde{P}_1 = (2, 5/2], \tilde{P}_2 = (1, 5/3], \dim M = 96,$$
$$m = (6, 3) \Rightarrow \quad \tilde{P}_1 = (3, 7/2], \tilde{P}_2 = (1, 5/3], \dim M = 88,$$

and all $P_i = \emptyset$. Therefore $p \in [0, 4]$. Use $\nu(p) = p - q = p - \sum_{p>\tilde{P}_k} l_k$, which can only lie in $\{0, 1\}$ in the case of a determinantal formula. This is equivalent to $\nu(p) = p - \sum_{pd_k>m_k+l_k} l_k$, hence this becomes $\nu(p) = p - \phi(2p > m_1 + 1)1 - \phi(3p > 5)2$, where $\phi(E) \in \{0, 1\}$ according to whether E is false or true. The possible values for p are $0, 1, 2, 3$; respectively $\nu = 0, 1, 0, 0$ or $1, 1$ and $q = 0, 0, l_2, l_1 + l_2$ or $l_2, l_1 + l_2$. The disjunction for $p = 3$ corresponds to 2 m-vectors. $\square$

We wish now to get a bound for those determinantal vectors for which some P_k is empty. Let $[\cdot]_k \in \{0, 1, \dots, d_k - 1\}$ denote remainder after division by d_k.

Definition 3.3. *Given $m \in \mathbb{Z}^r$ and $k \in \{1, \ldots, r\}$, define new vectors $m', m'' \in \mathbb{Z}^r$ whose j-th coordinates equal those of m $\forall j \neq k$ and s.t. $m'_k = m_k + d_k - [m_k]_k - 1 \geq m_k \geq m''_k = m_k - [m_k + l_k]_k$.*

Lemma 3.4. *The new vectors m', m'' differ from m at their k-th coordinate if $P_k(m) = \emptyset$.*

Lemma 3.5. *If $m \in \mathbb{Z}^r$ with $P_k(m) = \emptyset$ and $H^0(\mathbb{P}^{l_k}, m_k - pd_k) = 0$ (resp. $H^{l_k}(\mathbb{P}^{l_k}, m_k - pd_k) = 0$), then $P_k(m') \neq \emptyset$ and $H^0(\mathbb{P}^{l_k}, m'_k - pd_k) = 0$ (resp. $H^{l_k}(\mathbb{P}^{l_k}, m'_k - pd_k) = 0$), where $m'_k = m_k + d_k - [m_k]_k - 1$ as in Def. 3.3.*

Lemma 3.6. *If $m \in \mathbb{Z}^r$ with $P_k(m) = \emptyset$ and $H^0(\mathbb{P}^{l_k}, m_k - pd_k) = 0$ (resp. $H^{l_k}(\mathbb{P}^{l_k}, m_k - pd_k) = 0$), then $P_k(m'') \neq \emptyset$ and $H^0(\mathbb{P}^{l_k}, m''_k - pd_k) = 0$ (resp. $H^{l_k}(\mathbb{P}^{l_k}, m''_k - pd_k) = 0$), where $m''_k = m_k - [m_k + l_k]_k$ as in Def. 3.3.*

Another choice for m'_k is $m_k + d_k - [m_k + l_k]_k$, implying $P_k \neq \emptyset$, but not preserving the zero cohomologies necessarily. Lemmata 3.5, 3.6 imply

Theorem 3.7. *For any determinantal $m \in \mathbb{Z}^r$, define vectors $m', m'' \in \mathbb{Z}^r$ as in Def. 3.3 which differ from m only at the k-th coordinates, $1 \leq k \leq r$, s.t. $P_k(m) = \emptyset$. Then $P_k(m') \neq \emptyset, P_k(m'') \neq \emptyset$ and both m', m'' are determinantal.*

Corollary 3.8. *For a determinantal $m \in \mathbb{Z}^r$ with $P_k(m) = \emptyset$ for some $k \in \{1, \ldots, r\}$, we have $0 \leq m_k \leq d_k(n+1) - l_k - 1$.*

Proof. Since m'_k, m''_k define $P_k(m') \neq \emptyset, P_k(m'') \neq \emptyset$, we can apply Lem. 3.1. We use the lower bound with m''_k because $m''_k < m_k < m'_k$. $P_k(m) = \emptyset \Rightarrow d_k > l_k$, so

$$m''_k = m_k - [m_k + l_k]_k \geq -l_k \;\Rightarrow\; m_k \geq [m_k + l_k]_k - l_k \geq 1 - l_k,$$

because $[m_k + l_k]_k \geq 1$ by the proof of Lem. 3.4. If $m_k < 0$, for $P_k(m)$ to be empty we need $m_k + l_k < 0 \Leftrightarrow m_k < -l_k$ which contradicts the derived lower bound; so $m_k \geq 0$. For the upper bound,

$$m'_k = m_k + d_k - [m_k]_k - 1 \leq d_k(n+1) - 1 \;\Rightarrow$$
$$m_k \leq d_k n + [m_k]_k \leq d_k(n+1) - l_k - 1;$$

the latter follows from $[m_k]_k < d_k - l_k$, (Lem. 3.4). $(m_k + l_k)/d_k \leq n + 1 - (1/d_k) < n + 1$ implies the inclusion of the half-open interval in $(0, n+1)$. The possible values for m_k form a non-empty set, since the lower bound is zero and the upper bound is $d_k(n+1) - l_k - 1 \geq d_k - 1 > 0$ since $d_k > l_k \geq 1$. $\square$

So, in fact, the real interval $\tilde{P}_k \subset (0, n+1)$.

Corollary 3.9. *For a determinantal $m \in \mathbb{Z}^r$ and $k \in [r]$, $\max\{-d_k, -l_k\} \leq m_k \leq d_k(n+1) - 1 + \min\{d_k - l_k, 0\}$.*

This implies there is a finite number of vectors to be tested in order to enumerate all possible determinantal m. This could also be deduced from the fact that the dimension of $K_0(m)$ equals the degree of the resultant. Cor. 3.9 gives a precise bound for the box in which to search algorithmically for all determinantal m, including those that are "pure" in the terminology of [18]. Our Maple implementation, along with examples, is presented in Sect. 7. If we take, e.g., $m = (4, 3)$ and $k = 1$ in Ex. 3.2, the bound $2.3 - 1 - 1 = 4$ given by the Cor. 3.8 is attained. The bounds in Lem. 3.1 can also be attained (see Ex. 4.5 continued in Sect. 7) hence Cor. 3.9 is tight. It is possible that some combination of the coordinates of m restricts the search space.

4. PURE SYLVESTER-TYPE FORMULAE

This section constructs rectangular matrices that have at least one maximal minor which is a nontrivial multiple of the sparse resultant. The matrices of interest are of pure Sylvester-type. This implies the complex is:

$$\cdots \to K_1(m) = H^0(X, m - d) \to K_0(m) = H^0(X, m) \to 0$$

In particular, $m_k \geq 0$ for all $k = 1, \ldots, r$. In general $K_2(m) \neq 0$; $K_2(m) = 0$ implies a determinantal formula, which happens when the defects vanish [15].

To have a Sylvester-type formula, a necessary condition is that $H^0(X, m - pd) \neq 0$ for $\nu = 0, 1$ and $p - \nu = 0$. This implies $H^0(\mathbb{P}^{l_k}, m_k - \nu d_k) \neq 0$ for all $k \in \{1, \ldots, r\}$. Moreover, we must have $H^{p-\nu}(X, m - pd) = 0$ for $p - \nu = \sum_{j \in J} l_j$ where J is any subset satisfying $\emptyset \neq J \subset \{1, \ldots, r\}$.

Lemma 4.1. *If $m \neq m' \in \mathbb{Z}^r$ yield a Sylvester-type matrix and $m'_k \geq m_k$ for all $k \in \{1, \ldots, r\}$ then, the Sylvester matrix associated to m' is strictly greater than the Sylvester matrix associated to m.*

Definition 4.2. *For each choice of a permutation $\pi : \{1, \ldots, r\} \to \{1, \ldots, r\}$, define the degree vector m^π by $m^\pi_k := \left(1 + \sum_{\pi(j) \geq \pi(k)} l_j\right) d_k - l_k,\; k = 1, \ldots, r$.*

These are defined in [15], yielding determinantal Sylvester formulae when all defects are zero.

Lemma 4.3. *If $m \in \mathbb{Z}^r$ yields a Sylvester-type matrix, then $\exists i \in \{1, \ldots, r\}$ s.t. $\forall \pi : [r] \to [r]$ verifying $\pi(i) = 1$ it holds that $m_i \geq m^\pi_i$. Moreover $H^0(\mathbb{P}^{l_i}, m_i - pd_i) \neq 0$ where $p \leq 1 + \sum_{\pi(j) \in J} l_j$, for any subset $J : \emptyset \neq J \subset \{2, \ldots, r\}$.*

Theorem 4.4. *A degree vector $m \in \mathbb{Z}^r$ gives a Sylvester-type matrix iff there exists a permutation π s.t. $m_j \geq m^\pi_j$ for $j = 1, \ldots, r$. Moreover, the smallest Sylvester matrix is attained among the vectors m^π.*

Proof. Consider the necessary condition that $K_1(m) = H^0(X, m - d)$. We prove the forward direction by induction on $k = 1, \ldots, r$: The base case $k = 1$ was proven in Lem. 4.3. The inductive hypothesis for $k \in \{1, \ldots, r - 1\}$

specifies which cohomologies vanish and which not, where $m_u \geq m_u^\pi$, $\pi(u) \leq k$. In particular, for all subsets J s.t.

$$\emptyset \neq J \subset \{1, \ldots, r\} \setminus \{1, \ldots, k\},\ p = 1 + \sum_{\pi(j) \in J} l_j,\ p_0 = p + l_v,$$

for some $v : \pi(v) \leq k$, we assume:

$$H^{l_u}\left(\mathbb{P}^{l_u}, m_u - p_0 d_u\right) = 0,\ H^0\left(\mathbb{P}^{l_u}, m_u - p d_u\right) \neq 0. \quad (2)$$

For the inductive step, we exploit the necessary condition that $H^{p-1}(m - pd) = 0$ for $p = 1 + \sum_{\pi(j)>k} l_j$. By (2), $\exists i : H^{l_i}\left(\mathbb{P}^{l_i}, m_i - pd_i\right) = 0 \Leftrightarrow m_i \geq pd_i - l_i = m_i^\pi$ where we define $\pi(i) = k + 1$. To complete the step, we show $H^0\left(\mathbb{P}^{l_u}, m_u - pd_u\right) \neq 0$ where $\pi(u) \leq k + 1$, $p = 1 + \sum_{\pi(j) \in J} l_j$, and any subset J s.t. $\emptyset \neq J \subset \{k+2, \ldots, r\}$. The non-vanishing of the cohomology is equivalent to $m_u \geq pd_u$. It suffices to prove $m_i^\pi \geq d_i(1 + \sum_{\pi(j)>k+1} l_j)$. By definition, this reduces to $-l_i + d_i l_i \geq 0 \Leftrightarrow d_i \geq 1$. The converse direction follows from analogous arguments as above. The claim on minimality follows from Lem. 4.1. $\square$

This gives an algorithm for finding the minimal Sylvester formulae by testing at most $r!$ vectors m^π, which is implemented in MAPLE (Sect. 7). To actually obtain the square submatrix whose determinant is divisible by the sparse resultant, it suffices to execute a rank test. These matrices exhibit quasi-Toeplitz structure, implying that asymptotic complexity is quasi-quadratic in the matrix dimension [9]. Observe $P_k(m^\pi) \neq \emptyset$ because $\exists p \in \mathbb{Z} : p = 1 + \sum_{\pi(j) \leq \pi(k)} l_j$ s.t. $m_k^\pi < d_k p = m_k^\pi + l_k$ for all k.

EXAMPLE 4.5. Let $l = (2, 1, 1)$, $d = (2, 2, 2)$; the degree of the resultant is 960. Let $\sigma = \pi^{-1}$ be the permutation inverse to π; then $m_{\sigma(k)}^\pi := \left(1 + \sum_{j \geq k} l_{\sigma(j)}\right) d_{\sigma(k)} - l_{\sigma(k)}$. Here is a list of the $6 = 3!$ degree vectors m^π, among which we find the smallest Sylvester matrix of row dimension 1080, whereas the sparse resultant's degree is 960. Also shown are the permutations σ and the corresponding matrix dimensions. The symmetry between the last two polynomials makes certain dimensions appear twice.

$m^\pi =$		$\sigma =$		
	$(8, 5, 3)$		$1, 2, 3$	1080×1120
	$(8, 3, 5)$		$1, 3, 2$	1080×1120
	$(6, 9, 3)$		$2, 1, 3$	1120×1200
	$(4, 9, 7)$		$2, 3, 1$	1200×1440
	$(6, 3, 9)$		$3, 1, 2$	1120×1200
	$(4, 7, 9)$		$3, 2, 1$	1200×1440

Our MAPLE program, discussed in Sect. 7, enumerates 81 purely Sylvester matrices, none of which is determinantal. All Sylvester matrices not shown here have dimensions 1260×1400 or larger. $\square$

The map $K_1(m) \to K_0(m)$ is surjective, i.e., the matrix has at least as many columns as rows. In searching for a minimal formula, we should reduce $\dim K_0(m)$, i.e., the number of rows, since this defines the degree of the extraneous factor in the determinant. It is an open question whether $\dim K_0(m)$ reduces iff $\dim K_1(m)$ reduces. In certain system solving applications, the extraneous factor simply leads to a superset of the common isolated roots, so it poses no limitation. Even if it vanishes identically, perturbation techniques yield a nontrivial projection operator [5].

5. PURE BÉZOUT-TYPE FORMULAE

DEFINITION 5.1. *A Weyman complex is of pure Bézout type if $K_{-1}(m) = 0, K_1(m) = H^{l_1+\cdots+l_r}(X, m - (n+1)d)$ and $K_0(m) = H^0(m)$.*

Weyman complexes of pure Bézout type correspond to generically surjective maps

$$H^{l_1+\cdots+l_r}(X, m - (n+1)d) \to H^0(m) \quad (3)$$

s.t. any maximal minor is a nontrivial multiple of the multihomogeneous resultant. In fact, we shall show that the only possible such formulae are determinantal (i.e. $K_2(m) = 0$). We shall exhibit the corresponding differential in terms of the Bezoutian and characterize the possible degree vectors. We show that there exists a pure Bézout-type formula iff there exists a pure Sylvester formula. Remark that the dimension of the matrix with pure Bézout coefficients equals the dimension of the Sylvester matrix divided by $n+1$. Now we can generalize results in [3, 14] (cf. Sect. 1.2).

THEOREM 5.2. *There exists a determinantal formula of pure Bézout type iff for all k either $l_k = 1$ or $d_k = 1$, i.e. all defects vanish.*

PROOF. Recall that for each p there exists at most one integer $j : H^j(X, m - pd) \neq 0$ [18]. In fact, let $A(p) := \{k : m_k - pd - k < -l_k\}$ and $B(p) := \{k : m_k - pd_k \geq 0\}$. Denote $j := \sum_{k \in A(p)} l_k$. Then $H^{j'}(X, m - pd) = 0$ for all $j' \neq j$ and $H^j(X, m - pd) \neq 0$ iff $A(p) \cup B(p) = \{1, \ldots, r\}$.

There exists a pure Bézout-like formula for some degree vector m iff $\forall p \neq 0, n+1$ all cohomologies of $m - pd$ vanish. Then, for any $p \in \{1, \ldots, l\}$, $\exists k : k \notin A(p) \cup B(p)$, i.e., $p \in P_k(m)$. Then, $\{1, \ldots, l\} \subseteq \cup_{k=1}^r P_k(m)$ and so

$$l_1 + \ldots + l_r \leq \sum_{k=1}^r \# P_k(m) \leq \sum_{k=1}^r \lceil \frac{l_k}{d_k} \rceil.$$

Since $l_k \geq \lceil \frac{l_k}{d_k} \rceil$ for all k, we deduce that $l_k = \lceil \frac{l_k}{d_k} \rceil$, and this can only happen iff $l_k = 1$ or $d_k = 1$. $\square$

Let us study degree vectors yielding pure Bézout formulae.

DEFINITION 5.3. *For each choice of a permutation $\pi : \{1, \ldots, r\} \to \{1, \ldots, r\}$, let us define a degree vector*

$$m_k^\pi := -l_k + d_k \sum_{\pi(j) \geq \pi(k)} l_j, \quad k = 1, \ldots, r.$$

(3) implies $H^{l_j}\left(\mathbb{P}^{l_j}, m_j - (n+1)d_j\right) \neq 0$, $H^0\left(\mathbb{P}^{l_j}, m_j\right) \neq 0$.

LEMMA 5.4. *The existence of any pure Bézout formula implies $0 \leq m_j < (n+1)d_j - l_j$, for all j.*

In fact, the m^π of Def. 5.3 satisfy these constraints $\forall \pi$.

LEMMA 5.5. *If $m \in \mathbb{Z}^r$ yields a pure Bézout-type complex, then $\exists\, i \in \{1,\dots,r\}$ s.t. for any permutation $\pi : [r] \to [r]$ with $\pi(i) = 1$, $m_i \geq m_i^\pi$ and*

$$H^{l_i}\left(\mathbb{P}^{l_i}, m_i - (q + l_i + \nu)d_i\right) = 0, \quad \nu = 0, -1,$$
$$H^0\left(\mathbb{P}^{l_i}, m_i - qd_i\right) \neq 0, \quad q = \sum_{j \in J} l_j,$$

for any $J \subset \{1,\dots,r\} \setminus \{i\}, J \neq \emptyset$.

LEMMA 5.6. *If $m \in \mathbb{Z}^r$ yields a pure Bézout-type complex, it is possible to find a permutation π s.t. the degree vector m verifies $m_i \geq m_i^\pi, \forall i$.*

THEOREM 5.7. *A pure Bézout and generically surjective formula exists for some vector m iff it equals m^π of Def. 5.3, for some permutation π, and all defects are zero.*

The condition $K_2(m) = 0$, which yields a square matrix, is obtained by the hypothesis of a pure Bézout and generically surjective formula; i.e., there is no rectangular surjective pure Bézout formula.

COROLLARY 5.8. *If a generically surjective formula is of pure Bézout type, then it is determinantal. Furthermore, for any permutation π, the matrix is of the same dimension, i.e.* $\dim K_0(m) = \deg R/(n+1)$.

5.1 Explicit Bézout-type formulae

DEFINITION 5.9. *For any permutation $\pi : [r] \to [r]$, define permutation $\pi' : [r] \to [r]$ by $\pi'(i) = r + 1 - \pi(i)$.*

LEMMA 5.10. *Assuming all defects are zero, $m^\pi + m^{\pi'} = \rho$ for any permutation $\pi : [r] \to [r]$, where $\rho \in \mathbb{N}^r$ is the critical vector of Def. 1.2.*

Denote by x_i (resp. x_{ij}) the i-th variable group (resp. the j-th variable in the group), $i \in [r], j = 0,\dots,l_i$. Introduce r new groups of variables y_i with the same cardinalities and denote by y_{ij} their variables.

Given a permutation π, let the associated *Bezoutian* be the polynomial $B^\pi(x,y)$ obtained as follows: First dehomogenize the polynomials by setting $x_{i0} = 1, i = 1,\dots,r$; the obtained polynomials are denoted by $f_0,\dots,f_n$. Second, construct the $(n+1) \times (n+1)$ matrix with j-th column corresponding to polynomial f_j, $j = 0,\dots,l$, and whose x_{ij} variables are gradually substituted, in successive rows, by each respective y_{ij} variable. This construction is named after Bézout or Dixon and is well-known in the literature, e.g. [1, 8]. A general entry is of the form

$$f_j(y_{\sigma(1)},\dots,y_{\sigma(k-1)},y_{\sigma(k)1},\dots,y_{\sigma(k)t}, \\ x_{\sigma(k)(t+1)},\dots,x_{\sigma(k)l_{\sigma(k)}},x_{\sigma(k+1)},\dots,x_{\sigma(r)}) \tag{4}$$

where $\sigma := \pi^{-1}, k = 0,\dots,r, t = 1,\dots,l_k$. There is a single first row for $k = 0$, containing all the polynomials in the x_{ij} variables, whereas the last row has the same polynomials with all variables substituted by the y_{ij}. All intermediate rows contain the polynomials in a subset of the x_{ij} variables, the rest having been substituted by each corresponding y_{ij}. The number of rows is $1 + \sum_{j \in [r]} l_j = 1 + n$. Lastly, in order to obtain $B^\pi(x,y)$, we divide the matrix determinant by

$$\prod_{i=1}^{r}\prod_{j=1}^{l_i}(x_{ij} - y_{ij}). \tag{5}$$

EXAMPLE 5.11. Let $l = (1,2), d = (2,1)$. If $\pi = (12), \pi' = (21)$, then $m^\pi = (5,0), m^{\pi'} = (1,1)$. For both degree vectors, the matrix dimension is 6. To obtain $B^\pi(x,y)$ we construct a 4×4 matrix whose j-th column contains $f_j(x_{11},x_2), f_j(y_{11},x_2), f_j(y_{11},y_{21},x_{22}), f_j(y_{11},y_2), j = 0,\dots,3$. Then $B^\pi(x,y)$ contains the following monomials in the x_i and y_i variables respectively, 6 in each set of variables: $1, x_1, x_{21}, x_{22}, x_1x_{21}, x_1x_{22},\ 1, y_1, y_1^2, y_1^3, y_1^4, y_1^5$. So the final matrix is indeed square of dimension 6. □

LEMMA 5.12. *Let $B^\pi(x,y) = \sum b_{\alpha\beta}x^\alpha y^\beta$ where $\alpha = (\alpha_{ij}), \beta = (\beta_{ij}) \in \mathbb{Z}^n$, $i = 1,\dots,r, j = 1,\dots,l_i$. Set $\alpha_i = \sum_{j=1,\dots,l_i}\alpha_{ij}$, $\beta_i = \sum_{j=1,\dots,l_i}\beta_{ij}$, $\forall \alpha,\beta$. Then, $0 \leq \alpha_i \leq m_i^{\pi'}, 0 \leq \beta_i \leq m_i^\pi$ and $0 \leq \alpha_i + \beta_i \leq \rho_i$, $i = 1,\dots,r$.*

For generic polynomials, the upper bounds of α_i, β_i are attained. The lemma thus gives tight bounds on the support of the Bezoutian.

THEOREM 5.13. *Assume all defects are zero and $B^\pi(x,y)$ is defined as above. For any π, $(b_{\alpha\beta})$ is a square matrix of dimension*

$$\dim K_0(m) = \binom{l}{l_1,\dots,l_r} d_1^{l_1}\cdots d_r^{l_r} = \frac{\deg R}{(n+1)}.$$

Furthermore, $\det(b_{\alpha\beta}) = R(f_0,\dots,f_n)$.

PROOF. First, show that $(b_{\alpha\beta})$ is square of the desired size: But the dimensions are given by the number of exponent vectors α,β bounded by Lem. 5.12 which are exactly $\dim K_0(m^{\pi'}), \dim K_0(m^\pi)$ respectively. Both $m^\pi, m^{\pi'}$ are determinantal, hence both of these numbers are equal to $\deg R/(n+1)$, by Thm. 5.7 and Cor. 5.8. $R(f_0,\dots,f_n)$ divides every nonzero maximal minor of the matrix $(b_{\alpha\beta})$; cf. [1], [8, thm 3.13]. Since any nonzero proper minor has degree $< \deg R$, the determinant of the matrix $(b_{\alpha\beta})$ is nonzero and equals the resultant. □

6. EXAMPLE: A HYBRID DETERMINANTAL FORMULA

Assume $l = (3,2), d = (2,3)$. We present explicit formulae which can be extrapolated in general, giving an answer to the problem stated in [18, p. 578]. We plan to carry this extensively in a future work, but we include here the example as a hint for the interested reader. Our program enumerates 30 determinantal vectors m, with minimal matrix dimension 1320 achieved at $m = (6,3)$ and $(2,12)$. In both cases, $P_2(m) = \emptyset$, whereas $P_1(6,3) = \{4\}$ and $P_1(2,12) = \{2\}$.

This shows that the minimum matrix dimension may occur for some empty P_k, contrary to what one may think.

Moreover, the degree of the sparse resultant is $6\binom{5}{3,2}2^3 3^2 = 4320$. Since 1320 does not divide 4320, the minimal matrix is not of pure Bézout type; it is not of pure Sylvester type either. To specify the cohomologies and the linear maps that make the matrix formula explicit we compute, for degree vector $m = (6,3)$ and $p = 1,\ldots,6$ the different values of $m - pd$: $(4,0)$, $(2,-3)$, $(0,-6)$, $(-2,-9)$, $(-4,-12)$, $(-6,-15)$. The complex becomes $K_2 = 0 \to K_1 \to K_0 \to K_{-1} = 0$, with nonzero part

$$H^0(4,0)^{\binom{6}{1}} \oplus H^2(0,-6)^{\binom{6}{3}} \oplus H^5(-6,-15)^{\binom{6}{6}} \to$$
$$\to H^0(6,3)^{\binom{6}{0}} \oplus H^2(2,-3)^{\binom{6}{2}} \oplus H^5(-4,-12)^{\binom{6}{5}},$$

where we omitted the reference to the space $X = \mathbb{P}^3 \times \mathbb{P}^2$ in the notation of the cohomologies. Then $\dim K_1 = 210 + 200 + 910 = 1320 = 840 + 150 + 330 = \dim K_0$. By a slight abuse of notation, let $\delta_{\alpha,\beta}$ stand for the restriction of the above map to $H^\alpha \to H^\beta$. Then $\delta_{02} = \delta_{05} = \delta_{25} = 0$ by [18, prop. 2.5] and it suffices to study the maps below, of which the first 3 are of pure Sylvester type by [18, prop. 2.6] and the last 3 are of pure Bézout type as those of Sect. 5. These maps can be simplified using the dual cohomologies:

$$H^j(\mathbb{P}^{l_k}, m_k - pd_k) = H^{l_k - j}(\mathbb{P}^{l_k}, (\rho_k - m_k) - (n+1-p)d_k)^*,$$

where ρ is the critical vector of Def. 1.2. So, we have maps

$$\begin{array}{ll}
\delta_{00}: & H^0(4,0)^6 \to H^0(6,3)\\
\delta_{22}: & (H^0(0)\otimes H^0(3)^*)^{\binom{6}{3}} \to (H^0(2)\otimes H^0(0)^*)^{\binom{6}{2}}\\
\delta_{55}: & H^0(2,12)^* \to (H^0(0,9)^*)^6\\
\delta_{20}: & (H^0(0)\otimes H^0(3)^*)^{\binom{6}{3}} \to H^0(6,3)\\
\delta_{50}: & H^0(2,12)^* \to H^0(6,3)\\
\delta_{52}: & H^0(2,12)^* \to (H^0(2,0)^*)^{\binom{6}{2}}
\end{array}$$

The resultant matrix (of the previous map in the natural monomial bases) has the following aspect, indicated by the row and column dimensions:

$$\begin{array}{c} \\ 210 \\ 220 \\ 910 \end{array}
\begin{array}{c} \begin{array}{ccc} 840 & 150 & 330 \end{array} \\ \left[\begin{array}{ccc} \delta_{00} & 0 & 0 \\ \delta_{20} & \delta_{22} & 0 \\ \delta_{50} & \delta_{52} & \delta_{55} \end{array}\right] \end{array}
= \left[\begin{array}{ccc} S_{00} & 0 & 0 \\ B_{20}^{x_2} & \delta_{22} & 0 \\ B_{50} & B_{52}^{x_1} & S_{55}^T \end{array}\right]$$

where $S_{ij}, B_{ij}, B_{ij}^{x_k}$ stand for pure Sylvester and Bézout blocks, the latter coming from a Bezoutian with respect to variables x_k for $k = 1,2$, and S_{55}^T represents a transposed Sylvester matrix, corresponding to the dual of the Sylvester map $H^0(0,9)^6 \to H^0(2,12)$.

Let us take a closer look at δ_{22}, which denotes both the map and the corresponding matrix. Let $\alpha \in \mathbb{N}^{l_1}, \beta \in \mathbb{N}^{l_2}$, be the degree vectors of the elements of $H^0(2), H^0(3)^*$ respectively, thus $|\alpha| \le 2, |\beta| \le 3$. Let $I, J \subset \{0,\ldots,5\}, |I| = 3, |J| = 2$ express the chosen polynomials according to the homology exponents. Then the entries are given by

$$\delta_{22}(x_1^\alpha \otimes T_J, 1 \otimes S_I^\beta) = \begin{cases} 0, \text{ if } J \not\subset I, \\ \text{coef}(f_k) \text{ of } x_1^\alpha x_2^\beta, \text{ if } I \setminus J = \{k\}, \end{cases}$$

where $T_J \in H^0(0)^*, S_I^\beta$ are elements of the respective dual bases of monomials. Now take the Bézout maps: The matrix entries are given in (4) for $\sigma = (2,1)$: the entry (i,j), $i,j \in \{0,\ldots,5\}$ contains $f_j(x^{(1)}, \ldots, x^{(5-i)}, y^{(6-i)}, \ldots, y^{(5)})$, where each $x^{(i)}$ is a leading subsequence of x_{11}, x_{12}, x_{13}, x_{21}, x_{22}; similarly with the new variables $y^{(i)}$. The degree of the determinant, i.e. the Bezoutian, is $6,3,2,12$ in x_1, x_2, y_1, y_2 respectively and these coefficients fill in the matrix B_{50}. For the Bézout block $B_{52}^{x_1}$, consider "partial" Bezoutians defined from the 6 polynomials with the exception of those indexed in J, where J, I are as above. Only the x_1 variables are substituted by new ones, thus yielding a 4×4 matrix. For $B_{20}^{x_2}$, take all polynomials indexed in I and develop the Bezoutian with new variables y_2 from a 3×3 matrix. Hence the entries of the Bézout blocks have, respectively, degree $6,4,3$ in the coefficients of the f_i.

routine	function
`allDetVecs`	enumerate all determinantal formulae
`allsums`	compute all possible sums of the l_i's adding to $q \in \{0,\ldots,\sum_{i=1}^r l_i\}$.
`coHzero`	test whether $H^q(m - pd)$ vanishes
`coHdim`	compute the dimension of $H^q(m - pd)$
`dimKv`	compute the dimension of K_ν i.e. of the corresponding matrix
`findBez`	find all m-vectors yielding a pure Bézout-type formula; may choose to consider only determinantal formulae
`findSyl`	find all m-vectors yielding a pure Sylvester-type formula
`hasdeterm`	test whether a determinantal formula exists

Table 1: The main functionalities of our software.

7. IMPLEMENTATION

We have implemented on MAPLE V routines for the above operations, including those in Table 1. They are available in file `mhomo.mpl` on http://www-sop.inria.fr/galaad/-logiciels/emiris/soft_alg.html and illustrated below.

EXAMPLE 3.2 (CONT'D) Let $m = (4,3)$:

```
> Ns:=vector([1,2]):  Ds:=vector([2,3]):
> summs:=allsums(Ns):
> hasdeterm(Ns,Ds,vector([4,3]),summs);
                         true
> dimKv(Ns,Ds,vector([4,3]),summs,1);
                          96
> dimKv(Ns,Ds,vector([4,3]),summs,0);
                          96
```

□

EXAMPLE 4.5 (CONT'D) The MAPLE session first computes all 81 pure Sylvester formula by searching the appropriate range of 246 vectors. The smallest formulae are shown.

```
> Ns:=vector([2,1,1]):Ds:=vector([2,2,2]):
> allSyl:=findSyl(Ns,Ds):
Search of degree vecs from [4,3,3] to [8,9,9].
First array [4,7,9]:  dimK1=1440, dimK0=1200,
dimK(-1)=0(should be 0).
          #pure-Sylvester degree vectors= 81
tried 246, got 81 pure-Sylv formulae [m,dimK1,dimK0]:
```

```
> sort(convert(%,list),sort_fnc);
```

$$[[8,\ 5,\ 3,\ 1120,\ 1080],\ [8,\ 3,\ 5,\ 1120,\ 1080],\ [6,\ 9,\ 3,\ 1200,\ 1120],\ [6,\ 3,\ 9,\ 1200,\ 1120],\ [4,\ 9,\ 7,\ 1440,\ 1200],\ [4,\ 7,\ 9,\ 1440,\ 1200],\ [8,\ 6,\ 3,\ 1400,\ 1260],\ [8,\ 3,\ 6,\ 1400,\ 1260],\ \ldots]$$

```
> allDetVecs(Ns,Ds):
> allmsrtd := sort(convert(%,list),sort_fnc);
```

$\mathit{From},\ [-4,\ -3,\ -3],\ \mathit{to},\ [9,\ 10,\ 10],\ \mathit{start\ at},\ [-4,\ -3,\ -3]$

```
Tested 1452 m-vectors:  assuming Pk's nonempty.
Found 488 det'l m-vecs, listed with matrix dim:
```

$$\mathit{allmsrtd} := [[6,3,1,\ 224],\ [6,1,3,\ 224],\ [1,7,5,\ 224],\ [1,5,7,\ 224],\ [3,7,1,\ 240],\ [4,7,1,\ 240],\ [3,1,7,\ 240],\ [4,1,7,\ 240],\ [6,3,0,\ 262],\ [6,0,3,\ 262],\ [1,8,5,\ 262],\ [1,5,8,\ 262],\ \ldots]$$

```
> for i from 1 to nops( allmsrtd ) do print (
allmsrtd[i],Pksets(Ns,Ds,vector([allmsrtd[i][1],
allmsrtd[i][2],allmsrtd[i][3]]))):  od:
```

$$\begin{array}{c}
[6,\ 3,\ 1,\ 224],\ [[4,\ 4],\ [2,\ 2],\ [1,\ 1]]\\
[6,\ 1,\ 3,\ 224],\ [[4,\ 4],\ [1,\ 1],\ [2,\ 2]]\\
[1,\ 7,\ 5,\ 224],\ [[1,\ 1],\ [4,\ 4],\ [3,\ 3]]\\
[1,\ 5,\ 7,\ 224],\ [[1,\ 1],\ [3,\ 3],\ [4,\ 4]]\\
[3,\ 7,\ 1,\ 240],\ [[2,\ 2],\ [4,\ 4],\ [1,\ 1]]\\
[4,\ 7,\ 1,\ 240],\ [[3,\ 3],\ [4,\ 4],\ [1,\ 1]]\\
[3,\ 1,\ 7,\ 240],\ [[2,\ 2],\ [1,\ 1],\ [4,\ 4]]\\
[4,\ 1,\ 7,\ 240],\ [[3,\ 3],\ [1,\ 1],\ [4,\ 4]]\\
[6,\ 3,\ 0,\ 262],\ [[4,\ 4],\ [2,\ 2],\ [0,\ \mathit{NO_INT},\ 0]]\\
\ldots
\end{array}$$

The last 2 commands find all 488 determinantal vectors. The smallest formulae are indicated (the minimum dimension is 224) and for some we report the P_k's. No determinantal formulae is pure Sylvester. Notice that the assumption of empty sets P_k is used only in order to bound the search, but within the appropriate range empty P_k's are considered, so no valid degree vector is missed. This is illustrated by the last $P_3 = \emptyset$ marked *NO_INT*. □

Example 5.11 (Cont'd) The only pure Bézout formulae are the 2 determinantal formulae of Ex. 5.11, for which we have $m^\pi = (5,0), m^{\pi'} = (1,1)$.

```
> Ns:=vector([1,2]):Ds:=vector([2,1]):
> summs:=allsums(Ns):
> findBez(Ns,Ds,true); #not only determinantal
```

$\mathit{low} - \mathit{upper\ bounds},\ \mathit{1st\ candidate}:,\ [0,\ 0],\ [6,\ 1],\ [0,\ 0]$

```
Searched degree m-vecs for ANY pure Bezout formula.
Tested 15, found 2 pure-Bezout [m,dimK0,dimK1]:
```

$$\{[5,\ 0,\ 6,\ 6],\ [1,\ 1,\ 6,\ 6]\}$$

The search examined 15 degree vectors between the shown bounds. It is clear that both vectors are determinantal because the matrix dimensions are for both 6×6. □

8. FURTHER WORK

Our results can be generalized to polynomials with scaled supports or with a different degree d per polynomial. We plan to complete the description of hybrid determinantal formulae. Another question is whether the vectors m', m'' of Def. 3.3 lead to smaller or larger matrices than m. Notice that certain cohomologies, which were nonzero for m, may vanish for m' or m''. A problem related to the Sylvester formulae calls for improved algorithms for constraining the search of m, and for identifying in advance the nonzero maximal minor in the matrix, which leads to finding a determinant with exact degree in some polynomial. We are working on finding an explicit generic basis of the quotient by n polynomials of given degree $d \in \mathbb{Z}^r$.

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APPENDIX

PROOF OF LEM. 3.1 By [18, lem. 3.3(b)], $P_k \subset [0, n+1] \Rightarrow m_k/d_k \geq -1$ and $(m_k + l_k)/d_k \geq 0$, which imply the lower bound. Also, $(m_k+l_k)/d_k < n+2 \Leftrightarrow m_k \leq (n+2)d_k - l_k - 1$ and $m_k/d_k < n+1 \Leftrightarrow m_k \leq (n+1)d_k - 1$ yield the upper bound. Notice that the possible values for m_k form a non-empty set, since the 2 bounds are negative and positive respectively. □

PROOF OF LEM. 3.4 Let us write $m_k = jd_k + [m_k]_k$ for $j \in \mathbb{Z}$ so $\frac{m_k}{d_k} = j + \frac{[m_k]_k}{d_k}$. $P_k = \emptyset \Rightarrow \frac{m_k + l_k}{d_k} < j+1 \Rightarrow jd_k + [m_k]_k + l_k < (j+1)d_k$, so $[m_k]_k \leq d_k - l_k - 1 \Rightarrow l_k \leq -[m_k]_k + d_k - 1$ and thus $1 \leq d_k - [m_k]_k - 1$. Second, $[m_k + l_k]_k \geq 1$ because $[m_k + l_k]_k = 0 \Rightarrow (m_k + l_k)/d_k \in P_k(m)$. □

PROOF OF LEM. 3.5 Write $m_k = jd_k + [m_k]_k$ for some integer $j \in \mathbb{Z}$. To prove $P_k(m') \neq \emptyset$ we show $j+1 \in P_k(m')$, i.e.

$$\frac{m'_k}{d_k} < j+1 \leq \frac{m'_k + l_k}{d_k} \Leftrightarrow m'_k < (j+1)d_k \leq m'_k + l_k \Leftrightarrow$$
$$jd_k + d_k - 1 < (j+1)d_k \leq jd_k + d_k - 1 + l_k$$

which is clearly true since $1 \leq l_k$. $H^{l_k}(\mathbb{P}^{l_k}, m_k - pd_k) = 0 \Leftrightarrow m_k + l_k \geq pd_k$ hence $m'_k + l_k \geq pd_k$ because $m'_k \geq m_k$. $H^0(\mathbb{P}^{l_k}, m_k - pd_k) = 0 \Leftrightarrow m_k < pd_k$ so $j \leq p-1$. Hence, $m_k - [m_k]_k = jd_k \leq (p-1)d_k \Leftrightarrow m_k - [m_k]_k + d_k \leq pd_k$ which is the desired conclusion. By [18], $P_k(m') \subset [0, n+1]$ from which $j \in \{-1, 0, \ldots, l\}$. □

PROOF OF LEM. 3.6 Write $m_k + l_k = jd_k + [m_k + l_k]_k$ for some integer $j \geq 0$. To prove $P_k(m''_k) \neq \emptyset$ we show it contains j, i.e.,

$$\frac{m_k - [m_k + l_k]_k}{d_k} < j \leq \frac{m_k - [m_k + l_k]_k + l_k}{d_k} \Leftrightarrow$$
$$m_k - [m_k + l_k]_k < jd_k \leq m_k - [m_k + l_k]_k + l_k$$
$$\Leftrightarrow jd_k - l_k < jd_k \leq jd_k,$$

which is clearly true since $0 < l_k$. $H^0(m_k - pd_k) = 0 \Leftrightarrow m_k < pd_k \Rightarrow m''_k < pd_k$ because $m''_k \leq m_k$, hence $H^0(m''_k - pd_k) = 0$. $H^{l_k}(m_k - pd_k) = 0 \Leftrightarrow m_k + l_k \geq pd_k$, then $j \geq p$. Hence $m_k + l_k - [m_k + l_k]_k \geq pd_k$ which finishes the proof. By [18], $P_k(m'') \subset [0, n+1]$ hence $j \in \{0, \ldots, n+1\}$. □

PROOF OF LEM. 4.1 We must show $\dim K_0(m') \geq \dim K_0(m)$, i.e. $\dim H^0(m'_k - pd_k) \geq \dim H^0(m_k - pd_k)$ for $k \in \{1, \ldots, r\}$, i.e. $\binom{m'_k - pd_k + l_k}{l}_k \geq \binom{m_k - pd_k + l_k}{l}_k$. The cohomology is nonzero, thus $m'_k - pd_k \geq m_k - pd_k \geq 0$ and this implies the desired inequality because $\binom{s + l_k}{l}_k = (s + l_k) \cdots (s+1)/l_k!$. The inequality is strict since $\exists k : m'_k > m_k$. □

PROOF OF LEM. 4.3 For $p = n+1, \nu = 1$, a necessary condition is that $H^n(X, m - (n+1)d) = 0$. Hence $\exists\, i \in [r] : H^{l_i}(\mathbb{P}^{l_i}, m_i - (n+1)d_i) = 0 \Leftrightarrow m_i - (n+1)d_i \geq -l_i \Leftrightarrow m_i \geq m_i^\pi$ by choosing $\pi(i) = 1$. For any p as in the statement, $H^0(\mathbb{P}^{l_i}, m_i - pd_i) \neq 0 \Leftrightarrow m_i \geq pd_i$. Since $m_i - (n+1)d_i \geq -l_i$ it suffices to prove $(n+1)d_i - l_i \geq d_i(1 + \sum_{j \neq i} l_j) \geq d_i p$. The latter inequality is obvious $\forall p$, whereas the former reduces to $l_i d_i \geq l_i$ which holds since $d_i \geq 1$. □

PROOF OF LEM. 5.6 We use induction; the base case follows from Lem. 5.5. The inductive hypothesis, for $k \in \{1, \ldots, r-1\}$, is: $\exists U \subset \{1, \ldots, r\}$, $|U| = k$, s.t. $\pi(u) \leq k$, $m_u \geq m_u^\pi, \forall u \in U$ and

$$H^{l_u}\left(\mathbb{P}^{l_u}, m_u - (q + l_u + \nu)d_u\right) = 0, \quad \nu = 0, -1, \tag{6}$$
$$H^0\left(\mathbb{P}^{l_u}, m_u - qd_u\right) \neq 0, \quad q = \sum_{j \in J} l_j,$$

$\forall J \subset \{1, \ldots, r\} \setminus U, J \neq \emptyset$. Now the inductive step: The hypothesis on K_0 implies $H^p(X, m - pd) = 0$ for $p = \sum_{j \notin U} l_j$. Considering the inequality in (6) for $q = p$, $\exists i \in [r] \setminus U$ s.t. $H^{l_i}\left(\mathbb{P}^{l_i}, m_i - pd_i\right) = 0 \Leftrightarrow m_i + l_i \geq pd_i$ i.e. $m_i \geq m_i^\pi$ for $\pi(i) = k+1$ because $j \notin U \Leftrightarrow \pi(j) \geq k+1$. It suffices now to extend (6) for $q' = \sum_{j \in J'} l_j$ where $\emptyset \neq J' \subset [r] \setminus (U \cup \{i\})$. First, $m_i + l_i \geq pd_i \geq (q' + l_i)d_i$ implies the equations below. Second, $m_i \geq -l_i + pd_i = (p - l_i)d_i + l_i(d_i - 1) \geq q'd_i$ yields the inequality, so

$$H^{l_i}\left(\mathbb{P}^{l_i}, m_i - (q' + l_i + \nu)d_i\right) = 0, \ \nu = 0, -1.$$
$$H^0\left(\mathbb{P}^{l_i}, m_i - q'd_i\right) \neq 0.$$

m satisfies the hypothesis $K_{-1} = 0$ by (7) for $\nu = -1$ because every summand in K_{-1} contains some cohomology as in (7). Since $p \geq 0 \Rightarrow q = p - \nu \geq 1$ no summand has only zero cohomologies. By Lem. 5.4 and (7) for $\nu = 0$, m gives $K_0(m) = H^0(X, m)$ because all $H^{l_i} = 0$. □

PROOF OF THM. 5.7 It suffices to consider $K_1 = H^n(m - (n+1)d)$; it is nonzero by Lem. 5.4. We prove by induction that $m = m^\pi$ by using the fact that all other summands in (1) for K_1 vanish. For $H^0(m - d)$ to vanish, there must exist $i \in [r]$: $H^0(\mathbb{P}^{l_i}, m_i - d_i) = 0 \Leftrightarrow m_i < d_i$. Hence we need to define $\pi(i) = r$ because $\pi(i) < r \Rightarrow m_i^\pi \geq -l_i + d_i(l_i + 1) = d_i + l_i(d_i - 1) \geq d_i$. Moreover, $m_i^\pi = -l_i + d_i l_i < d_i \Leftrightarrow \delta_i = 0$ by Lem. 1.3.

There is a unique integer in $[m_i^\pi, d_i)$ because $m_i^\pi + 1 \geq d_i \Leftrightarrow -l_i + d_i l_i + 1 \geq d_i \Leftrightarrow (l_i - 1)(d_i - 1) \geq 0$. Hence $m_i = d_i - 1 < d_i(q+1)$ for any $q \geq 0$, therefore $H^0(m_i - (q+1)d_i) = 0$. Furthermore, for $q \geq l_i$, $H^{l_i}(m_i - (1+q)d_i) \neq 0 \Leftrightarrow m_i + l_i < (q+1)d_i \Leftrightarrow l_i - 1 < qd_i$ which holds. This proves the inductive basis. The inductive hypothesis is: $\forall\, u \in U \subset [r]$, where $|U| = k$, $\pi(u) > r - k$, then $\delta_u = 0$, $m_u = m_u^\pi$ and

$$H^0(m_u - (1 + \sum_{\pi(j) < \pi(u)} l_j)d_u) = 0 \neq H^{l_u}(m_u - (1 + \sum_{j \in J} l_j)d_u), \tag{7}$$

$\forall J : U \subset J \subset [r]$. For the inductive step, consider that $H^q(X, m - (1+q)d)$ must vanish for $q = \sum_{j \in U} l_j$. None of its summand cohomologies $H^{l_u}(m_u - (1+q)d_u)$ vanish due to the last inequality. So $\exists i : H^0(m_i - (1+q)d_i) = 0 \Leftrightarrow m_i < (1+q)d_i$.

Hence $\pi(i) = r - k$ so that $m_i = m_i^\pi = -l_i + d_i \sum_{\pi(j) \geq r-k} l_j < (1+q)d_i \Leftrightarrow -l_i + d_i l_i < d_i \Leftrightarrow \delta_i = 0$ by Lem. 1.3. No larger m_i works because m_i^π is the maximum integer strictly smaller than $(1+q)d_i$. And $\pi(i) < r - k$ would make m_i too large. Now extend the inequality (7) to J' where $(U \cup \{i\}) \subset J'$ and observe $m_i^\pi < d_i \sum_{\pi(j) \geq r-k} l_j < d_i(1 + \sum_{j \in J'} l_j)$.

The hypothesis is proven $\forall U \subset [r]$, including the case $|U| = r$. For the converse, assume $\exists \pi : m = m^\pi$ and all defects vanish. Then K_0, K_1 satisfy all conditions for a pure Bézout formula. Furthermore, $K_{-1} = 0$, hence the formula is generically surjective. □

PROOF OF LEM. 5.10 $m_i^\pi + m_i^{\pi'} = d_i(n + l_i) - 2l_i, \forall i$ because the sum in the parenthesis includes $\{l_j : \pi(j) \geq \pi(i)\} \cup \{l_j : \pi'(j) \geq \pi'(i)\}$, and latter set is $\{l_j : \pi(j) \leq \pi(i)\}$. So $m_i^\pi + m_i^{\pi'} = d_i l_i - l_i + \rho_i - d_i + 1 = \rho_i + d_i(l_i - 1) - (l_i - 1) = \rho_i + (l_i - 1)(d_i - 1) = \rho_i$ because of the zero defects. □

PROOF OF LEM. 5.12 By Lem. 5.10 it suffices to bound α_i, β_i. But α_i is the degree of the x_i in the determinant decreased by l_i in order to account for the division by (5). The former equals the product of d_i with the number of rows where an x_{ij} variable appears for any $j \in [1, l_i]$. These are the first row, the rows where y_j are introduced for $j \in \{\sigma(1), \ldots, \sigma(k-1)\}$ s.t. $\sigma(k) = i$, and another $l_i - 1$ rows when $\sigma(k) = i$. The condition on j : $\pi(j) < \pi(i)$ is equivalent to $r + 1 - \pi(j) > r + 1 - \pi(i)$, hence $\alpha_i \leq -l_i + d_i \sum_{\pi'(j) \geq \pi'(i)} l_j = m_i^{\pi'}$. Similarly, we prove the upper bound on β_i. The rows containing y_{ij} for some $j \in [1, l_i]$ are those where $j \in \{\sigma(k+1), \ldots, \sigma(r)\} : \sigma(k) = i$, another $l_i - 1$ rows when $\sigma(k) = i$, and the last row. Now, $\pi(j) \geq k + 1 > k = \pi(i)$, so $\beta_i \leq -l_i + d_i \sum_{\pi(j) \geq \pi(i)} l_j = m_i^\pi$. Clearly $\alpha_i, \beta_i \geq 0$. □