Implicitization of rational hypersurfaces and syzygies II

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Recalling the problem

$$\phi : \mathbf{P}^{n-1} \dashrightarrow \mathbf{P}^{n},$$

$f := (f_0, \ldots, f_n), f_i \in R := k[X_1, \ldots, X_n]$ homogeneous of degree $d \geq 1$, such that the closure of its image is a hypersurface $\mathcal{H}$.

Goal: Compute the equation H of $\mathcal{H}$.

- $I := (f_0, \ldots, f_n) \subset R$ be the ideal generated by the f_i's,
- $X := \mathrm{Proj}(R/I) \subset \mathbf{P}^{n-1}$ be the subscheme defined by I, i.e. the base points of ϕ.

If $\Gamma_0 \subset \mathbf{P}^{n-1} \times \mathbf{P}^n$ is the graph of $\phi : (\mathbf{P}^{n-1} - X) \longrightarrow \mathbf{P}^n$ and Γ the Zariski closure of Γ_0, one has :

$$\mathcal{H} = \overline{\phi(\mathbf{P}^{n-1} - X)} = \overline{\pi(\Gamma_0)} = \pi(\Gamma)$$

$$\Gamma = \mathrm{Proj}(\mathcal{R}_I)$$

with

$$\mathcal{R}_I := R \oplus I \oplus I^2 \oplus \cdots = R \oplus It \oplus I^2t^2 \oplus \cdots$$

$\mathcal{R}_I$ is the Rees algebra associated to I.

If $\mathfrak{P} := \ker(s)$, $\mathfrak{P}_1$ (the degree 1 part of $\mathfrak{P}$) is the module of syzygies of the f_i's :

$$a_0T_0 + \cdots + a_nT_n \in \mathfrak{P}_1 \iff a_0f_0 + \cdots + a_nf_n = 0.$$

By definition, $\mathcal{S}_I := \mathrm{Sym}_R(I)$ is the symmetric algebra associated to I. Let $V := \mathrm{Proj}(\mathcal{S}_I)$ the associated variety. We have natural onto maps

$$S \longrightarrow S/\langle \mathfrak{P}_1 \rangle \qquad \text{and} \qquad \mathcal{S}_I \simeq S/\langle \mathfrak{P}_1 \rangle \longrightarrow S/\mathfrak{P} \simeq \mathcal{R}_I$$

which correspond to the embeddings

$$\Gamma \subseteq V \subset \mathbf{P}^{n-1} \times \mathbf{P}^n.$$

Theorem

If $\dim X = 0$, $\Gamma = V$ if and only if X is locally a complete intersection.

In fact, $\mathfrak{P} = \langle \mathfrak{P}_1 \rangle^{sat}$.

Approximation complexes

Proposition ([Busé- Jouanolou '02])

Assume that $\Gamma = V$ *and let* η *be such that* $H^0_{\mathfrak{m}}(\mathcal{S}_I)_\mu = 0$ *for all* $\mu \geq \eta$. *Then,*

$$\mathrm{ann}_{k[T_0,\dots,T_n]}(\mathcal{S}_I^{\eta}) = \mathfrak{P} \cap k[T_0,\dots,T_n] = \langle H \rangle.$$

So, we can use graded pieces of a resolution of $\mathcal{S}_I$ to compute H.

$\mathcal{S}_I$ has a known *canonical* resolution via the approximation complexes introduced and studied by Herzog, Simis and Vasconcelos.

In particular, the $\mathcal{Z}$-complex, constructed by means of two Koszul complexes, gives a representation matrix for $\mathcal{H}$ which *generalizes what we saw for rational curves*, once one gets a bound for the exponent η in the Proposition.

Approximation complexes

Let $y = (y_1, \ldots, y_r)$ be an r-tuple of elements in a ring A. The (homological) Koszul complex $K_\bullet(y;A)$ is the complex with modules $K_p(y;A) := \bigwedge^p A^r \simeq A^{\binom{p}{r}}$ and maps $d_p : K_p(y;A) \longrightarrow K_{p-1}(y;A)$

$$g.e_{i_1} \wedge \cdots \wedge e_{i_p} \longmapsto g.\sum_{j=1}^{p}(-1)^{j+1} y_{i_j} e_{i_1} \wedge \cdots \widehat{\wedge e_{i_j}} \cdots \wedge e_{i_p}.$$

$Z_p(y;A) := \ker(d_p)$ are the cycles and $H_p(y;A) := Z_p(y;A)/\mathrm{im}(d_{p+1})$ the homologies.

We consider $f_i \in R \subset S$ as elements of S and the two complexes $K_\bullet(f;S)$ and $K_\bullet(T;S)$ where $T := (T_0, \ldots, T_n)$.
These complexes have the **same** modules $K_p = \bigwedge^p S^{n+1} \simeq S^{\binom{p}{n+1}}$ and differentials $d_\bullet^f$ and $d_\bullet^T$

Approximation complexes

It directly follows from the definitions that $d^f_{p-1} \circ d^T_p + d^T_{p-1} \circ d^f_p = 0$, so that $d^T_p(Z_p(f;S)) \subset Z_{p-1}(f;S)$.
The complex $\mathcal{Z}_\bullet := (Z_\bullet(f;S), d^T_\bullet)$ is the called $\mathcal{Z}$-complex associated to the f_i's.

• Notice that $Z_p(f;S) = S \otimes_R Z_p(f;R)$ and

— $Z_0(f;R) = R$,

— $Z_1(f;R) = \mathrm{Syz}_R(f_0,\ldots,f_n)$,

— the map $d^T_1 : S \otimes_R \mathrm{Syz}_R(f_0,\ldots,f_n) \longrightarrow S$ is defined by :

$$(a_0,\ldots,a_n) \longmapsto a_0T_0 + \cdots + a_nT_n.$$

We will look at graded pieces *in the variables we want to eliminate*.

Aproximation complexes

The cokernel of the last map (in the piece of degree μ) is $\mathcal{S}_I^{\mu}$.

Theorem

If X is lci and $H^0_{\mathfrak{m}}(\mathcal{S}_I)_\mu = 0$, then the gcd of the maximal minors of any matrix representing this map equals $H^{\deg(\phi)}$.

What is a representation matrix?

- A rational surface $\mathscr{S}$ is given as the closed image of a map

$$\begin{array}{ccc} \mathbb{A}^2 & \overset{f}{\dashrightarrow} & \mathbb{P}^3 \\ s & \mapsto & (f_0(s):f_1(s):f_2(s):f_3(s)) \end{array}$$

  where the f_i are polynomials such that $\gcd(f_0,\ldots,f_3)=1$.

- A matrix representation M of $\mathscr{S}$ is a matrix with entries in $\mathbb{K}[T_0,T_1,T_2,T_3]$, generically of full rank, such that the rank of $M(P)$ drops iff the point $P\in\mathbb{P}^3$ lies on $\mathscr{S}$.
- The greatest common divisor of all minors of M of maximal size equals $F^{\deg(f)}$.

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Main tools

Linear syzygies

Given polynomials $f_0, \dots, f_3 \in \mathbb{K}[s_1, s_2]$, a syzygy on $f_0, \dots, f_3$ is a 4-tuple of polynomials $(h_0, \dots, h_3)$ such that $\sum_{i=0}^{3} h_i f_i = 0$.

Dealing with monomials: toric embeddings

We set one variable X_i for each integer point p_i in Q and we record multiplicatively (by binomial equations), the affine relations among these points. We then translate $f_i(s)$ to $g_i(X)$.

The method distilled in simplest terms

Basic general algorithm [BDD] sparse, [BCJ] classic

- **INPUT:** $(f_0(s), f_1(s), f_2(s), f_3(s))$ with Newton polytopes contained in P (a lattice polygon in the first quadrant), satisfying suitable hypotheses.
- **STEP 1:** Consider syzygies $(h_0, \ldots, h_3)$ with $N(h_i) \subset 2P = \{p_1 + p_2, p_i \in P\}$. Let $(h_0^{(j)}, , \ldots, h_3^{(j)}), j = 1, \ldots, N$, be a $\mathbb{K}$-basis of such syzygies.
- **STEP 2:** Represent the syzygies as linear forms $L_j = h_0^{(j)} T_0 + \cdots + h_3^{(j)} T_3$. Write $h_i^{(j)} = \sum_{\beta \in 2P \cap \mathbb{Z}^2} h_{i,\beta}^{(j)} s^\beta$ and switch:

$$L_j = \sum_i h_i^{(j)} T_i = \sum_\beta \left(\sum_i h_{i,\beta}^{(j)} T_i \right) s^\beta.$$

- **OUTPUT:** The matrix M of linear forms $\ell_{j,\beta} := \sum_i h_{i,\beta}^{(j)} T_i$.

First step

- We follow joint work with N. Botbol and M. Dohm, further extended to higher dimensions and to different codomains by Botbol.
- Surface parametrization of $\mathscr{S}$ given by

$$\begin{array}{ccc} \mathbb{A}^2 & \stackrel{f}{\dashrightarrow} & \mathbb{P}^3 \\ (s_1, s_2) & \mapsto & (f_0 : f_1 : f_2 : f_3)(s_1, s_2) \end{array}$$

  where $f_i \in \mathbb{K}[s,t]$ are polynomials such that $\gcd(f_0, \dots, f_3) = 1$ and $\mathbb{K}$ is a field (e.g. $\mathbb{Q}, \mathbb{R}$).
- First step: extend f to a map $\mathscr{T} \dashrightarrow \mathbb{P}^3$ for a suitable algebraic compactification $\mathscr{T}$ where f is densely defined.

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The toric embedding in naive terms

Example

Assume Q is the unit square, with 4 integer points:

$$p_0 = (0,0), p_1 = (1,0), p_2 = (0,1), p_3 = (1,1).$$

A polynomial f with Newton polytope Q looks like

$$f(s_1, s_2) = a_{(0,0)} + a_{(1,0)}\, s_1 + a_{(0,1)}\, s_2 + a_{(1,1)}\, s_1 s_2.$$

We take 4 new variables $X_0, \dots, X_3$ together with $X_0X_3 - X_1X_2 = 0 \rightleftharpoons$ $p_0 + p_3 = p_1 + p_2$. and so to the multiplicative relation $s^{p_0} s^{p_3} = s^{p_1} s^{p_2}$ between the monomials with these exponents.
The associated variety $\mathscr{T}$ lies in $\mathbb{P}^{4-1}$ with coordinate ring $A = \mathbb{K}[X_0, \dots, X_3]/\langle X_0X_3 - X_1X_2 \rangle$. f gets translated to:

$$g(X_0, \dots, X_3) = a_{(0,0)}\, X_0 + a_{(1,0)}\, X_1 + a_{(0,1)}\, X_2 + a_{(1,1)}\, X_3,$$

with the relation $X_0X_3 - X_1X_2 = 0$.

Choosing a compactification suitable for f

- $\mathrm{N}(f) \subset \mathbb{R}^2$ the Newton polytope of $f_0, \ldots, f_3$, that is, the smallest integer polygon which contains the exponents of all the monomials that occur in some f_i.
- $\mathrm{N}'(f)$ the smallest contraction of $\mathrm{N}(f)$ with integer vertices (i.e. $d \cdot \mathrm{N}'(f) = \mathrm{N}(f)$ for $d \in \mathbb{N}$). E.g., if $\mathrm{N}(f)$ is the triangle of size d, $\mathrm{N}'(f)$ is the unit triangle.
- $\mathrm{N}'(f)$ determines a projective toric variety $\mathscr{T} \subseteq \mathbb{P}^m$ as the closed image of the embedding

$$\begin{array}{ccc} \mathbb{A}^2 & \stackrel{\rho}{\dashrightarrow} & \mathbb{P}^m \\ (s_1, s_2) & \mapsto & (\ldots : s_1^i s_2^j : \ldots) \end{array}$$

  where $(i,j) \in \mathrm{N}'(f) \cap \mathbb{Z}^2$.
- Actually: Any polytope Q with $\mathrm{N}(f) \subseteq d \cdot Q$ for some d could work as well...

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Toric embeddings

- f factorizes through $\mathscr{T}$ in the following way

$$\begin{array}{ccc} \mathbb{A}^2 & \overset{f}{\dashrightarrow} & \mathbb{P}^3 \\ {\scriptstyle \rho}\downarrow & \nearrow{\scriptstyle g} & \\ \mathscr{T} & & \end{array}$$

- New homogeneous parametrization $g = (g_0 : g_1 : g_2 : g_3)$ with $g_i \in A = \mathbb{K}[X_0, \ldots, X_m]/I(\mathscr{T})$ and $deg(g_i) = d$, generating an ideal $I \subset A$.
- $\mathbb{P}^2$ (Q = unit triangle) and $\mathbb{P}^1 \times \mathbb{P}^1$ (Q = unit square) are special cases.
- Main difficulty: working over the *Cohen-Macaulay* affine normal semigroup ring A instead of a polynomial ring

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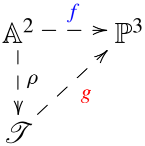

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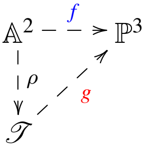

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Hypotheses

Theorem

$H^0_{\mathfrak{m}}(\mathrm{Sym}_A(I))_\nu = 0$ for all $\nu \geq \nu_0 = 2\mathrm{d} - \alpha$, where $\alpha \leq 2$ is the biggest integer such that $\alpha \cdot \mathrm{N}'(f)$ has **no** *interior lattice point.*
This bound can be improved in the presence of base points.

So, here are the hypotheses we need:

Theorem (BDD)

Suppose that there are only finitely many isolated base points of g which are a local complete intersection.
Then for all $\nu \geq 2d - \alpha$ the first matrix M_ν of $(\mathcal{Z}_\bullet)_\nu$ is a matrix representation of $\mathscr{S}$.
As we said above, the bound can be improved depending on the base points.

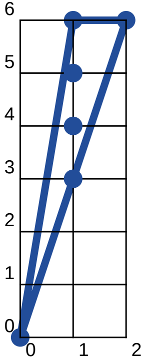
Testing the hypotheses?

We'd need to check whether there are only finitely many isolated base points of g which are a local complete intersection.

- This can be checked in particular cases.
- But what if we don't check this and run the algorithm? . . . nothing bad!
- We just check whether M_ν has full rank.
  - If the rank is not maximal, then there is a base point p which is not an almost local complete intersection.
  - If the rank is maximal, it may happen that the rank of M_ν drops at some other places besides $\mathscr{S}$ due to the existence of a non complete intersection base point.

The method into action

- Consider the parametrization with 6 monomials:
  $(f_0, f_1, f_2, f_3) = (2 + s^2t^6, st^6 + 2, st^5 - 3st^3, st^4 + 5s^2t^6)$
- $\mathrm{N}(f) =$



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- Coordinate ring $A = \mathbb{K}[X_0, \ldots, X_5]/J$, where $J = (X_3^2 - X_2X_4, X_2X_3 - X_1X_4, X_2^2 - X_1X_3, X_1^2 - X_0X_5)$
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- For $\nu_0 = 2d = 2$ the matrix M_{ν_0} is a matrix representation of size 17×34.
- The method fails over $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ due to non-LCI base points!

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The method into action

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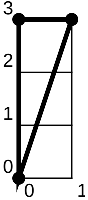
Is $\mathrm{N}'(f)$ always the optimal choice?

- Previous example with polytope Q:



- $\mathrm{N}(f) \subset 2 \cdot Q$, so the parametrization factorizes through the toric variety associated to Q.

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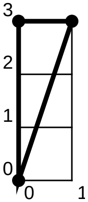

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  over the coordinate ring
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- **Philosophy:** compromise between two criteria:
  - polytope should be small (higher degree d)
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Use implicitization.m2

Check all 6 assertions in the example with the M2 code in:

`http://mate.dm.uba.ar/~ nbotbol/Implicitization.m2`

Choosing Q

How to choose the best polygon?

What if bad base points?

How to read the information in M_{ν_0} even if it is **not** of full rank ?

Improving the M2 algorithm without a toric embedding

[Botbol'10] shows how to improve the algorithm working with the multihomogeneous Cox ring of the toric variety instead of using any projective embedding.

Design ad-hoc good strategies to compute the syzygies (structured linear algebra) in the sparse case, without explicitly considering the toric embedding or the multihomogeneous setting.

Can we improve the sizes/ performance in the case of unmixed systems (i.e. the input polynomials may have different monomial supports)?

Integrating prediction of structure and homologic tools

How to integrate the prediction of the Newton polytope of the implicit equation (via convex geometry/intersection theory/tropical tools) with these homological methods? How to predict particular coefficients without needing the whole computation?

Inputs

– If you have Macaulay2 in your PC, just:

– download the file “Implicitization.m2” from http://mate.dm.uba.ar/~nbotbol/Implicitization.m2

– run M2

– copy the code below

```
-- Exercise code
S = QQ[s,t];
f0 = 2 + s^2 * t^6
f1 = s * t^6 + 2
f2 = s * t^5 - 3 * s * t^3
f3 = s * t^4 + 5 * s^2 * t^6
--
needsPackage ``Polyhedra'';
load ``Implicitization.m2'';
L = {f0, f1, f2, f3}
```

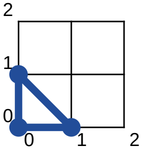
```
latticePoints polynomialsToPolytope L
tEmb=newToricEmbedding L
A= teToricRing tEmb
describe A
```

```
g = teToricRationalMap L;
```

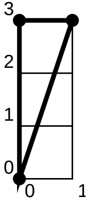
Also checking the rank drop

```
rM=representationMatrix (teToricRationalMap L,2);
R = QQ[s,t,X_0..X_3];
Eq = sub(sub(rM,R),{X_0 => (sub(f0,R)),X_1 => (sub(f1,R)),
        X_2 => (sub(f2,R)),X_3 => (sub(f3,R))});
rank rM
rank Eq
```

We get a warning that the rank is not maximal




```
Q2=convexHull matrix{{0,0,1},{0,1,0}}
gQ2 = teToricRationalMap (L,Q2)
rMQ2=representationMatrix (gQ2,14);
R = QQ[s,t,X_0..X_3];
EqQ2 = sub(sub(rMQ2,R),{X_0 => (sub(f0,R)),X_1 => (sub(f1,R)),
        X_2 => (sub(f2,R)),X_3 => (sub(f3,R))});
rank rMQ2
rank EqQ2
```




```
Q1=convexHull matrix{{0,0,1},{0,3,3}}
gQ1 = teToricRationalMap (L,Q1)
tEmb1=newToricEmbedding L
A1= teToricRing tEmb1
describe A1
g = teToricRationalMap L;
```

And the implicit equation again

```
rMQ1=representationMatrix (gQ1,2);
R = QQ[s,t,X_0..X_3];
EqQ1 = sub(sub(rMQ1,R),{X_0 => (sub(f0,R)),X_1 => (sub(f1,R)),
       X_2 => (sub(f2,R)),X_3 => (sub(f3,R))});
rank rMQ1
rank EqQ1
iEq1 = implicitEq (L,2,Q1)
(factor iEq1)#2#0
```

```
iEq = implicitEq (L,2)
(factor iEq)#2#0
```

```
ring r = 0,(t,x1,x2,x3,s,u), dp;
poly f0 = s^2 + s^3 * t;
poly f1 = s^3 * t^6 + 1;
poly f2 = s * t^2 + 2 * s^3 * t5;
poly f3 = s^2 + s^3 * t^6;
ideal i = f1 - x1 * f0, f2 - x2 * f0, f3 - x3 * f0, u * f0 - 1;
list k = eliminate(i, s*t*u);
k;
```

Another example of Singular code for elimination

```
ring r = 0,(x,y,s,t,a,b,c,d),dp;
ideal i = ax + by, cx + dy, (xs - 1) * (ty - 1);
ideal j = std(i);
ideal k = eliminate(j, xyst);
k;
ad - bc.
```