Implicitization of rational hypersurfaces and syzygies I

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KTH, May 2, 2011

The problem we want to solve

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- *Let f be a rational parametrization of a hypersurface $\mathscr{S} = (F = 0) \subset \mathbb{A}^3$, $F \in \mathbb{K}[T_1, T_2, T_3]$.*

$$\begin{array}{ccc} \mathbb{A}^2 & \stackrel{f}{\dashrightarrow} & \mathbb{A}^3 \\ s = (s_1, s_2) & \mapsto & \left(\frac{f_1(s)}{f_0(s)}, \frac{f_2(s)}{f_0(s)}, \frac{f_3(s)}{f_0(s)}\right) \end{array}$$

- $f_i \in \mathbb{K}[s_1, s_2]$ *with* $\gcd(f_0, \dots, f_3) = 1$ *and* $F(f(s)) = 0$ *(whenever defined).*
- **We assume the parametrization f known but the implicit equation F not known!**
- *That, is, we want to switch from parametric to implicit representations of rational surfaces.*

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Some basic notations/facts

- A dashed arrow $\dashrightarrow$ means that the map might not be defined everywhere.
- F is defined up to multiplicative constant
- We will always assume that the dimension of $\mathscr{S} = (F = 0)$ is 2 (equivalently, for almost all $p = f(s)$ in the image of f, the number of preimages by f is finite).
- This number is called the degree of f and noted $\deg(f)$.
- We can instead consider the map $\tilde{f} : \mathbb{A}^2 \dashrightarrow \mathbb{P}^3$ with image inside 3-dimensional projective space given by

$$s \mapsto (f_0(s) : f_1(s) : f_2(s) : f_3(s)).$$

- f and $\tilde{f}$ carry essentially the same information.
- Or we can consider the rational parametrization from another algebraic variety (other than $\mathbb{A}^2$) which contains the domain of f as a dense subset.

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Predicting the monomials that could occur in F

- The convex hull of the exponents of the monomials ocurring in a non zero polynomial h is called the Newton polytope $\mathrm{N}(h)$ of h.
- When h is a polynomial in s_1, s_2 of degree (at most) d, its Newton polytope $\mathrm{N}(F)$ is (contained in) the triangle with vertices $(0,0), (d,0), (0,d)$. Its Euclidean area is $d^2/2$ and its lattice area is $2\,d^2/2 = d^2$.

Theorem (Classic)

For generic polynomials $f_0, \ldots, f_3$ of degree d, the degree of F is d^2 and its Newton polytope is the triangle with vertices $(0,0,0), (d^2,0,0), (0,d^2,0), (0,0,d^2)$.

Theorem ([Sturmfels- Jie-Yai Yu'94])

For generic polynomials $f_0, \ldots, f_3$ with fixed Newton polytope P, the degree of F is the lattice area a of P and its Newton polytope is the triangle with vertices $(0,0,0), (a,0,0), (0,a,0), (0,0,a)$.

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First naive algorithm

- Assume the Newton polytope $\mathrm{N}(F)$ of F is known (as in the previous theorems) and number $m_1, \ldots, m_N \in \mathbb{N}^3$ the integer (lattice) points in $\mathrm{N}(F)$.
- Consider indeterminates $c = (c_1, \ldots, c_N)$ and write $F = \sum_{i=1}^{N} c_i T^{m_i}$.
- Substitute $T = f(s)$ and equate to 0 the coefficient of each power of s_1, s_2 that occurs.
- This sets a system of linear equations in c, with solution space of dimension 1. **Any nonzero solution c will give a choice of implicit equation F.**

This solves the problem, but … which is the size of this linear system?

For instance, in case f_i are generic polynomials of degree d in (s_1, s_2), the number of unknowns is $\binom{d^2+3}{3}$ ($\approx d^6/6$) and the number of equations is $\binom{d^3+2}{2}$ ($\approx d^6/2$).

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Implicitize $T - f(s) = 0$ means eliminating the s-variables

General strategies to eliminate variables

- Reduce the problem to a linear algebra problem
- Hide the variables one wants to eliminate in the (monomial) basis.
- Use determinants.

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A minimal incomplete history

- The use of syzygies for the implicitization of (conic) surfaces goes back to Steiner in 1832.
- Meyer in 1887 described syzygies of 3 polynomials and made a general conjecture proved by Hilbert in 1890.
- In 1995, Sederberg and Chen reintroduced the use of syzygies, by a method termed as "Moving curves and surfaces".
- Cox realized they were using syzygies. Several papers with other coauthors (Busé, Chen, D'Andrea, Goldman, Sederberg, Zhang).
- Jouanolou and Busé in 2002 abstracted, generalized on a sound basis the method of Sederberg-Chen via approximation complexes, a tool in homological commutative algebra that had been developed by Herzog, Simis and Vasconcelos.

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Motivation: Matrix representations for curves

- A planar rational curve $\mathscr{C}$ over a field $\mathbb{K}$ is given as the image of a map

$$\begin{array}{ccc} \mathbb{P}^1 & \overset{f}{\dashrightarrow} & \mathbb{P}^2 \\ s & \mapsto & (f_0(s) : f_1(s) : f_2(s)), \end{array}$$

  $f_i \in \mathbb{K}[s]$ homogeneous polynomials of degree d in s, $gcd(f_0, f_1, f_2) = 1$.

- A (linear) **syzygy** can be represented as a linear form $L = h_0 T_0 + h_1 T_1 + h_2 T_2$ in the new variables $T = (T_0, T_1, T_2)$ with $h_i \in \mathbb{K}[s]$ such that

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- $\mathrm{Syz}(\phi)$ = { all linear syzygies}. For $\nu \in \mathbb{N}$, the graded part $\mathrm{Syz}(\phi)_\nu$ $(\deg(h_i) \leq \nu)$ is a $\mathbb{K}$-vector space with dimension $N(\nu) < \infty$.
- **Attention: here comes the main elimination step:**
  Write for each syzygy $(h_0^i, \dots, h_3^i)$, $i = 1, \dots, N(\nu)$, in a basis:

$$L_i = L_i(s,T) = \sum_{j=0,1,2} h_j^i(s) T_j = \sum_{j=0,1,2} \left(\sum_{k=0}^{\nu} c_{jk}^i s_1^k s_2^{\nu-k}\right) T_j$$

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- Let M_ν be the $N(\nu) \times (\nu+1)$ matrix of coefficients of the L_i's with respect to a $\mathbb{K}$-basis of $\mathbb{K}[s]_\nu$:

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- If $\nu = d-1$, then M_ν is a square matrix, such that $\det(M_\nu) = F^{\deg(f)}$, where F is an implicit equation of $\mathscr{C}$.
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  where the f_i are polynomials such that $\gcd(f_0, \ldots, f_3) = 1$.
- A matrix representation M of $\mathscr{S}$ is a matrix with entries in $\mathbb{K}[T_0, T_1, T_2, T_3]$, generically of full rank, such that the rank of $M(P)$ drops iff the point $P \in \mathbb{P}^3$ lies on $\mathscr{S}$.
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- In general, the matrix M_ν of linear syzygies is not a square matrix representation for $\mathscr{S}$ (i.e. for no degree ν).
- Some special classes of surfaces (e.g. ruled surfaces, canal surfaces): square matrix representations exist.
- Two main approaches:
  - Use quadratic relations (i.e. linear syzygyes among the products f_if_j of two of the f_i) to construct square matrices. Sederberg, Cox, D'Andrea, Wang and others: methods to build matrix representations by means of linear and quadratic syzygies.
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Advantages

- Require only minimal assumptions on the parametrization
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Linear syzygies, quadratic syzygies, . . . , implicit equation

A common shape

- Linear syzygies of degree ν: $H(s,T) = \sum_{i=0}^{3} h_i(s)T_i$ such that $\sum_{i=0}^{3} h_i(s)f_i(s) = 0$. Thus, $\deg(H)$ in s variables is ν, $\deg(H)$ in T variables is 1.
- Quadratic syzygies of degree ν': $H(s,T) = \sum_{i\leq j=0}^{3} h_{i,j}(s)T_iT_j$ such that $\sum_{i,j=0}^{3} h_{i,j}(s)f_if_j(s) = 0$. Thus, $\deg(H)$ in s variables is ν', $\deg(H)$ in T variables is 2.
- Implicit equation (of degree D): $H(s,T) = \sum_{|\alpha|\leq D} h_\alpha T^\alpha$ such that $\sum_\alpha h_\alpha f^\alpha(s) = 0$. Thus, $\deg(H)$ in s variables is 0, $\deg(H)$ in T variables is D.
- So to go from linear syzygies to the implicit equation we play the game of lowering the degree in the s variables to 0 (which increases the degree in the T varibles up to (at most) D)!

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The method distilled in simplest terms

Basic general algorithm [BDD] sparse, [BCJ] classic

- **INPUT:** $(f_0(s), f_1(s), f_2(s), f_3(s))$ with Newton polytopes contained in P (a lattice polygon in the first quadrant), satisfying suitable hypotheses.
- **STEP 1:** Consider syzygies $(h_0, \ldots, h_3)$ with $N(h_i) \subset 2P = \{p_1 + p_2, p_i \in P\}$. Let $(h_0^{(j)}, \ldots, h_3^{(j)}), j = 1, \ldots, N$, be a $\mathbb{K}$-basis of such syzygies.
- **STEP 2:** Represent the syzygies as linear forms $L_j = h_0^{(j)} T_0 + \cdots + h_3^{(j)} T_3$. Write $h_i^{(j)} = \sum_{\beta \in 2P \cap \mathbb{Z}^2} h_{i,\beta}^{(j)} s^\beta$ and switch:

$$L_j = \sum_i h_i^{(j)} T_i = \sum_\beta \left(\sum_i h_{i,\beta}^{(j)} T_i \right) s^\beta.$$

- **OUTPUT:** The matrix M of linear forms $\ell_{j,\beta} := \sum_i h_{i,\beta}^{(j)} T_i$.

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- **STEP 1:** Consider syzygies $(h_0, \ldots, h_3)$ with $N(h_i) \subset 2P = \{p_1 + p_2, p_i \in P\}$. Let $(h_0^{(j)}, \ldots, h_3^{(j)}), j = 1, \ldots, N$, be a $\mathbb{K}$-basis of such syzygies.
- **STEP 2:** Represent the syzygies as linear forms $L_j = h_0^{(j)} T_0 + \cdots + h_3^{(j)} T_3$. Write $h_i^{(j)} = \sum_{\beta \in 2P \cap \mathbb{Z}^2} h_{i,\beta}^{(j)} s^\beta$ and switch:

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Size of the matrices in the algorithm

- Assume for example that P is the triangle of size d. Then, in fact, as we will see, it is enough to consider syzygies of degree $2d-2$. Therefore, to find them, we have a system on $4\binom{2d}{2}$ variables with $\binom{3d}{2}$ equations. That is, both sizes, as well as the vector space dimension of the space of syzygies in this degree, are quadratic in d.
- The matrix M has then a number of rows quadratic in d. The number of its columns equals $\binom{2d}{2}$, again quadratic in d.

The syzygy method is then a great improvement on the naive linear algebra method! (naive approach is of the order of d^6).

Remark

As we will see, in the case of curves, the Sylvester resultant matrix to find the implicit equation via a resultant computation, uses a matrix of size $2d$ for a parametrization with polynomials of degree d, while the syzygy method uses 2 matrices of size d, as the Bézout resultant.

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Plan for the rest of the class + afternoon class

What follows

- Theoretical tools
- Hypotheses
- Improvements
- Examples
- M2 Implementation.
- Some open questions.

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Recalling the problem

Let

$$\phi \ : \mathbf{P}^{n-1} \dashrightarrow \mathbf{P}^n,$$

be a rational map defined by $f := (f_0, \ldots, f_n)$, $f_i \in R := k[X_1, \ldots, X_n]$ homogeneous of degree $d \geq 1$, such that the closure of its image is a hypersurface $\mathcal{H}$.

Goal: Compute the equation H of $\mathcal{H}$.

We let :

- $I := (f_0, \ldots, f_n) \subset R$ be the ideal generated by the f_i's,
- $X := \mathrm{Proj}(R/I) \subset \mathbf{P}^{n-1}$ be the subscheme defined by I, i.e. the base points of ϕ.

If $\Gamma_0 \subset \mathbf{P}^{n-1} \times \mathbf{P}^n$ is the graph of $\phi : (\mathbf{P}^{n-1} - X) \longrightarrow \mathbf{P}^n$ and Γ the Zariski closure of Γ_0, one has :

$$\mathcal{H} = \overline{\phi(\mathbf{P}^{n-1} - X)} = \overline{\pi(\Gamma_0)} = \pi(\Gamma)$$

where $\pi : \mathbf{P}^{n-1} \times \mathbf{P}^n \longrightarrow \mathbf{P}^n$ is the second projection, and bar denotes the Zariski closure (or equivalently the closure for the usual topology in the case $k = \mathbf{C}$).

On the algebraic side, one has

$$\Gamma = \mathrm{Proj}(\mathcal{R}_I)$$

with

$$\mathcal{R}_I := R \oplus I \oplus I^2 \oplus \cdots = R \oplus It \oplus I^2t^2 \oplus \cdots$$

$\mathcal{R}_I$ is the Rees algebra associated to I.

The embedding $\Gamma \subset \mathbf{P}^{n-1} \times \mathbf{P}^n$ corresponds to the natural graded map :

$$\begin{array}{rcl} S := R[T_0, \ldots, T_n] & \xrightarrow{s} & \mathcal{R}_I \\ T_i & \mapsto & f_i t \in It = (\mathcal{R}_I)_1 \end{array}$$

Recall that $R := k[X_1, \ldots, X_n]$.

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If $\mathfrak{P} := \ker(s)$, $\mathfrak{P}_1$ (the degree 1 part of $\mathfrak{P}$) is the module of syzygies of the f_i's :

$$a_0T_0 + \cdots + a_nT_n \in \mathfrak{P}_1 \iff a_0f_0 + \cdots + a_nf_n = 0.$$

By definition, $\mathcal{S}_I := \mathrm{Sym}_R(I)$ is the symmetric algebra associated to I.
Let $V := \mathrm{Proj}(\mathcal{S}_I)$ the associated variety. We have natural onto maps

$$S \longrightarrow S/\langle \mathfrak{P}_1 \rangle \qquad \text{and} \qquad \mathcal{S}_I \simeq S/\langle \mathfrak{P}_1 \rangle \longrightarrow S/\mathfrak{P} \simeq \mathcal{R}_I$$

which correspond to the embeddings

$$\Gamma \subseteq V \subset \mathbf{P}^{n-1} \times \mathbf{P}^n.$$

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Recall that X is a local complete intersection if each local ideal at any zero of X can be generated by $n-1$ polynomials.

Theorem

If $\dim X = 0$, $\Gamma = V$ if and only if X is locally a complete intersection.

So, when X is locally a complete intersection (lci), we can use (linear) syzygies to compute H.
A more algebraic way to state the theorem is the following:

Theorem

If X is a lci, the prime ideal $\mathfrak{P}$ is the saturation of the ideal generated by its elements of degree 1 in the T_i's (the syzygies).

$\mathcal{R}_I = S/\mathfrak{P}$ is a domain, but $S_I = S/\langle \mathfrak{P}_1 \rangle$ has torsion.

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Saturation with respect to the maximal ideal

An ideal I in a polynomial ring $R := k[X_0, \ldots, X_n]$ is saturated (or, more precisely $\mathfrak{m}$-saturated) if $I : \mathfrak{m} = I$, where $\mathfrak{m} := (X_0, \ldots, X_n)$.

So, I is saturated if : $X_i f \in I, \forall i \Rightarrow f \in I$.

The ideal $I^{sat} := \bigcup_j (I : \mathfrak{m}^j)$ is saturated, it is the smallest saturated ideal containing I and is called the saturation of I.

For any R-module M, its module of 0-th local cohomology is $H^0_{\mathfrak{m}}(M) = \{m \in M / \mathfrak{m}^N m = 0, \text{ for some } N\}$.

$$I^{sat} = I + H^0_{\mathfrak{m}}(R/I), \quad R/I^{sat} = (R/I)/H^0_{\mathfrak{m}}(R/I).$$

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An overview of our second class

Proposition ([Busé- Jouanolou '02])

Assume that $\Gamma = V$ *and let* η *be such that* $H^0_{\mathfrak{m}}(\mathcal{S}_I)_\mu = 0$ *for all* $\mu \geq \eta$. *Then,*

$$\operatorname{ann}_{k[T_0,\ldots,T_n]}(\mathcal{S}_I^\eta) = \mathfrak{P} \cap k[T_0,\ldots,T_n] = \langle H \rangle.$$

So, we can use graded pieces of a resolution of $\mathcal{S}_I$ to compute H.

$\mathcal{S}_I$ has a known *canonical* resolution via the approximation complexes introduced and studied by Herzog, Simis and Vasconcelos.

In particular, the $\mathcal{Z}$-complex, constructed by means of two Koszul complexes, gives a representation matrix for $\mathcal{H}$ which *generalizes what we saw for rational curves*, once one gets a bound for the exponent η in the Proposition.

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So, we can use graded pieces of a resolution of $\mathcal{S}_I$ to compute H.

$\mathcal{S}_I$ has a known *canonical* resolution via the approximation complexes introduced and studied by Herzog, Simis and Vasconcelos.

In particular, the $\mathcal{Z}$-complex, constructed by means of two Koszul complexes, gives a representation matrix for $\mathcal{H}$ which *generalizes what we saw for rational curves*, once one gets a bound for the exponent η in the Proposition.

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