

Algebraic Geometry: Computations and Applications

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Homework # 9, due Monday, May 9

1. Consider the rational parametrization

$$\phi : \mathbf{P}^2 \dashrightarrow \mathbf{P}^3,$$

given by $f := (ac^2, b^2(a+c), ab(a+c), bc(a+c))$ with $R := \mathbf{Q}[a, b, c]$. The ring of the target will be $\mathbf{Q}[x, y, z, t]$. The ideal $\mathfrak{P} \subset S := R[x, y, z, t] = \mathbf{Q}[a, b, c, x, y, z, t]$ of the corresponding Rees algebra equals

$$\mathfrak{P} = \langle ay - bz, at - cz, bt - cy, act - b(a+c)x, bx(z+t) - at^2, xy(z+t) - zt^2 \rangle$$

- (a) Which is the implicit equation?
 - (b) Compute $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{P}_4$. Check that $\mathfrak{P}_2 \subset (\mathfrak{P}_1 : \langle a, b, c \rangle)$.
 - (c) Prove that the saturation of the base locus ideal $I = \langle f_0, \dots, f_3 \rangle$ is the complete intersection ideal $\langle ac^2, b(a+c) \rangle$
2. Consider again the parametrization of the first exercise, given by polynomials of degree $d = 3$. The degree of the implicit equation H is known to be given by the following formula:

$$\deg(H) \cdot \deg(\phi) = d^2 - \sum_{p \in V(I)} e_p, \quad (1)$$

where e_p is the Serre multiplicity at each base point.

Check that there are 3 base points p_1, p_2, p_3 which are local complete intersections. In this case, the Serre multiplicity e_{p_i} equals the vector space dimension of the corresponding local ring $(R/I)_{p_i}$. Compute these 3 multiplicities and check formula (1).

3. Consider the linear forms $\ell_1(s) = 2s_1 + s_2 + 3s_3$, $\ell_2(s) = -2s_1 - s_2 - 2s_3$, $\ell_3(s) = s_1 + s_2$, $\ell_4(s) = -s_1 - s_2 - s_3$. We form the parametrization $\psi : \mathbf{P}^2 \dashrightarrow \mathbf{P}^3$ as follows:

$$f_1 = \ell_1^2 \ell_3 \quad , \quad f_2 = \ell_1 \ell_2 \ell_3 \quad , \quad f_3 = \ell_1^3 \quad , \quad f_0 = \ell_2^2 \ell_4 .$$

The implicit equation of the closure of the image of ϕ is given by:

$$H(T) = T_0 T_1^2 T_3 + T_0^2 T_1 T_2 + T_0 T_1^3 + T_2^2 T_3^2 .$$

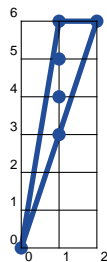
- Prove that there are only two base points $p = (-2 : 1 : 1)$, $p' = (1 : -2 : 0)$.
- Prove that p is a local complete intersection and compute its multiplicity.
- Prove that p' is not a local complete intersection and check that the dimension of the local quotient at p' equals 3.
- Compute $e_{p'}$ using formula (1).

4. (Optional) Download the M2 file `Implicitization.m2` from:

<http://mate.dm.uba.ar/~nbotbol/Implicitization.m2>

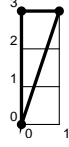
and check all 6 assertions in the following example. The instructions follow at the end. Also, play with other examples. If you have any doubts concerning the code, please write to Nicolás Botbol (nbotbol@dm.uba.ar)

- Consider the parametrization with 6 monomials: $(f_0, f_1, f_2, f_3) = (2 + s^2 t^6, st^6 + 2, st^5 - 3st^3, st^4 + 5s^2 t^6)$
- $N(f) =$



- $(f_0, f_1, f_2, f_3) = (2 + s^2 t^6, st^6 + 2, st^5 - 3st^3, st^4 + 5s^2 t^6)$

- Consider $Q = N(f)$.
- **Assertion 1:** Coordinate ring is $A = k[X_0, \dots, X_5]/J$, where $J = (X_3^2 - X_2X_4, X_2X_3 - X_1X_4, X_2^2 - X_1X_3, X_1^2 - X_0X_5)$
- **Assertion 2:** New parametrization g over \mathcal{T} is given by $(g_0, g_1, g_2, g_3) = (2X_0 + X_5, 2X_0 + X_4, -3X_1 + X_3, X_2 + 5X_5)$
- **Assertion 3:** For $\nu_0 = 2d = 2$ the matrix M_{ν_0} is a matrix representation of size 17×34 .
- **Assertion 4:** The method fails over \mathbb{P}^2 (i.e $Q =$ unit triangle.)
- Consider now the polytope Q :



- $N(f) \subset 2 \cdot Q$, so f factorizes through the toric variety ass. to Q .
- **Assertion 5:** New parametrization is defined by $(g_0, g_1, g_2, g_3) = (2X_0^2 + X_4^2, 2X_0^2 + X_3X_4, -3X_0X_4 + X_2X_4, X_1X_4 + 5X_4^2)$ over the coordinate ring $A = k[X_0, \dots, X_4]/(X_2^2 - X_1X_3, X_1X_2 - X_0X_3, X_1^2 - X_0X_2)$.
- **Assertion 6:** For $\nu_0 = 2$: we get a matrix representation of size 12×19 , compared to 17×34 for $N'(f)$.

Set $\nu_0 = 2d = 2$. The greatest common divisor of the 17-minors of the 17×34 matrix M_{ν_0} is the homogeneous implicit equation of the surface:

$$\begin{aligned}
& 2809T_0^2T_1^4 + 124002T_1^6 - 5618T_0^3T_1^2T_2 + 66816T_0T_1^4T_2 + 2809T_0^4T_2^2 \\
& - 50580T_0^2T_1^2T_2^2 + 86976T_1^4T_2^2 + 212T_0^3T_2^3 - 14210T_0T_1^2T_2^3 + 3078T_0^2T_2^4 \\
& + 13632T_1^2T_2^4 + 116T_0T_2^5 + 841T_2^6 + 14045T_0^3T_1^2T_3 - 169849T_0T_1^4T_3 \\
& - 14045T_0^4T_2T_3 + 261327T_0^2T_1^2T_2T_3 - 468288T_1^4T_2T_3 - 7208T_0^3T_2^2T_3 \\
& + 157155T_0T_1^2T_2^3T_3 - 31098T_0^2T_2^3T_3 - 129215T_1^2T_2^3T_3 - 4528T_0T_2^4T_3 \\
& - 12673T_2^5T_3 - 16695T_0^2T_1^2T_2^2 + 169600T_1^4T_2^2 + 30740T_0^3T_2T_2^2 \\
& - 433384T_0T_1^2T_2T_2^2 + 82434T_0^2T_2^2T_2^2 + 269745T_1^2T_2^2T_2^2 + 36696T_0T_2^3T_2^2 \\
& + 63946T_2^4T_2^2 + 2775T_0T_1^2T_2^3 - 19470T_0^2T_2T_2^4 + 177675T_1^2T_2T_2^3 \\
& - 85360T_0T_2^2T_2^3 - 109490T_2^3T_2^3 - 125T_1^2T_2^4 + 2900T_0T_2T_2^4 \\
& + 7325T_2^2T_2^4 - 125T_2T_2^5
\end{aligned}$$

Or set $X_0 = 1$ to get the affine equation.

M2 Code:

```

S=QQ[s,t];
f0=2+s^2*t^6;
f1=s*t^6+2;
f2=s*t^5-3*s*t^3;
f3=s*t^4+5*s^2*t^6;

needsPackage "Polyhedra"
load "Implicitization.m2"
L={f0,f1,f2,f3};
latticePoints polynomialsToPolytope L
tEmb=newToricEmbedding L
A= teToricRing tEmb
describe A
g = teToricRationalMap L
rM=representationMatrix (teToricRationalMap L,2);
R = QQ[s, t, X_0..X_3];
Eq = sub(sub(rM, R), {X_0 => (sub(f0, R)), X_1 => (sub(f1, R)),X_2 =>
(sub(f2, R)), X_3 => (sub(f3, R))}); rank rM
rank Eq

Q2=convexHull matrix{{0,0,1},{0,1,0}}
gQ2 = teToricRationalMap (L,Q2)
rMQ2=representationMatrix (gQ2,14);
R = QQ[s, t, X_0..X_3];
EqQ2 = sub(sub(rMQ2, R), {X_0 => (sub(f0, R)), X_1 => (sub(f1, R)),X_2 =>
(sub(f2, R)), X_3 => (sub(f3, R))});
rank rMQ2
rank EqQ2

Q1=convexHull matrix{{0,0,1},{0,3,3}}
gQ1 = teToricRationalMap (L,Q1)
tEmb1=newToricEmbedding L
A1= teToricRing tEmb1
describe A1
g = teToricRationalMap L;
rMQ1=representationMatrix (gQ1,2);

```

```
R=QQ[s, t, X_0..X_3];
EqQ1 = sub(sub(rMQ1, R), {X_0 => (sub(f0, R)), X_1 => (sub(f1, R)), X_2 =>
(sub(f2, R)), X_3 => (sub(f3, R))});
rank rMQ1
rank EqQ1
iEq1 = implicitEq (L,2,Q1)
(factor iEq1)#2#0
iEq = implicitEq (L,2)
(factor iEq)#2#0
```

Write to me if you want the instructions in a text file.