Algebraic Geometry: Computations and Applications

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Homework # 8, due Monday, May 2

1. Let $I = \langle f_1, \ldots, f_s \rangle \subset \mathbb{C}[x_1, \ldots, x_n]$ be a zero dimensional ideal and $g \in \mathbb{C}[x_1, \ldots, x_n]$. Denote by $A = \mathbb{C}[x_1, \ldots, x_n]/I$ the quotient ring.

   (i) Find an algorithm to decide if g "separates" the points of I, that is, if g takes different values at different zeros of I.

   (ii) Prove that the class $\overline{g}$ of g in A is invertible (with respect to the product) if and only if $g(p) \neq 0$ for any $p \in V_{\mathbb{C}}(I)$. If this is the case, propose an algorithm to find a polynomial h such that $\overline{h}\,\overline{g} = 1$.

2. Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be a zero dimensional ideal and denote by $h_i(T)$ the minimal polynomial of the multiplication map $m_{x_i} : A \to A$, where A denotes again the quotient by I.

   (i) Prove that $h_i(x_i)$ is the (monic) generator of $I \cap \mathbb{C}[x_i]$.

   (iii) Prove that I is radical if and only if the matrices of $m_{x_1}, \ldots, m_{x_m}$ (in any basis) are diagonalizable.

3. (i) Let $f \in \mathbb{R}[x]$ of degree $d > 0$. Prove that $V(f) + V(f(-x)) \leq d$ and deduce that if all roots of f are real, $V(f)$ equals the number of positive roots of f.

   (ii) Let $M \in \mathbb{R}^{m \times m}$ a symmetric matrix. Give an algorithm to compute the signature of A.

   (iii) Given a zero dimensional ideal $I \subset \mathbb{R}[x_1, x_2]$, propose an algorithm to compute the number of zeros of I in the positive orthant $(\mathbb{R}_{>0})^2$.

4. Let $I = \langle f_1, \dots, f_s \rangle \subset \mathbb{C}[x_1, \dots, x_n]$ and denote by $g = gcd(f_1, \dots, f_s)$ the greatest common divisor.

   (i) If $n = 1$, prove that $I = \langle g \rangle$ (so, $f_1, \dots, f_s$ have a common root if and only if $g \neq 1$).

   (ii) Let $n > 1$ and f_1, f_2 two coprime non constant polynomials ($g = 1$). Can $V(I)$ be empty?

   (iii) For $n = 1$, g can be obtained via Euclides algorithm (based on the division algorithm). Describe an algorithm to compute g for any n.

5. Let $I \subset \mathbb{C}[x_1, \dots, x_n]$ be a zero dimensional ideal with zeros $\{p_1, \dots, p_d\}$ in $\mathbb{C}^n$. Denote by $\mathfrak{M}_i$ the maximal ideal at p_i, that is, the ideal of all polynomials which vanish at p_i. Prove that there exists a natural number N such that $\cap_{i=1}^{d} \mathfrak{M}_i^N \subseteq I$.

6. Let $f_1, f_2 \in \mathbb{C}[x]$ without common roots, with $\deg(f_1) = d_1 \geq \deg(f_2) = d_2 > 0$. For any $k \in \mathbb{N}$, denote by S_k the $\mathbb{C}$-vector space of polynomials in $\mathbb{C}[x]$ of degree at most k. Take any $a \geq d_2$ and consider the map:

$$\varphi_f : S_{a-d_1} \times S_{a-d_2} \to S_a$$

   defined by $\varphi_f(g_1, g_2) = g_1 f_1 + f_2 g_2$.

   Describe all the values of a for which φ_f is injective, surjective or an isomorphism. In this last case, write down its matrix in the monomial bases.