Algebraic Geometry: Computations and Applications

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Homework # 10, due Monday, May 16

1. Let k be an algebraically closed field and $f \in k[x_1, \dots, x_n]$. Prove that f does not have any zero in $(k^*)^n$ if and only if f is a monomial.

2. Check the following assertions about the Wilkinson polynomial $W_{20} = \prod_{i=1}^{20}(x+i) = \sum_{j=0}^{20} c_j x^j$ in a computer algebra system:

   - $W_{20}(x) + 10^{-9}x^{19}$ has only 12 real roots. Check that the complex non real roots do not "seem" to have small imaginary part.
   - Compute $D(a,b)$ the discriminant of the family of polynomials $W(a,b,x) := W_{20}(x) + ax^{19} + bx^{18}$ and check that $(a,b) = (0,0)$ is a singular point.

3. Consider the following dehomogenization δ_3 of the discriminant Δ_3 of a generic univariate polynomial of degree 3:

$$\Delta_3 \;=\; 27\,x_1^2x_4^2 - 18\,x_1x_2x_3x_4 + 4\,x_1x_3^3 + 4\,x_2^3x_4 - x_2^2x_3^2,$$

$$\delta_3 \;=\; \Delta_3(1,1,y_1,y_2).$$

   Check that the vectors $(1,-2,1,0), (2,-3,0,1)$ give a basis of the integer kernel of A and consider the associated linear forms: $\ell_1(x,y) := u + 2v, \ell_2 := -2u - 3v$, $\ell_3 := u, \ell_4 := v$. The Horn-Kapranov parametrization of $\delta_3 = 0$ is then given by

$$(y_1, y_2) = (\ell_1\ell_3/\ell_2^2, \ell_1^2\ell_4/\ell_2^3).$$

   Check the validity of the recipe to find the Newton polytope of δ_3 given at the end of the class.

Check that, up to a monomial, $\Delta_3(x)$ equals $\delta_3(x_1x_3/x_2^2, x_1^2x_4/x_2^3)$ and explain why this happens.

4. Consider the parametrization of a rational curve $\mathcal{C}$ given by $y_1 = f_1(u,v), y_2 = f_2(u,v)$, where f_1, f_2 are rational functions of degree 0 (that is, $f_i = p_i/q_i$ with p_i, q_i polynomials with complex coefficients of the same degree, $q_i \neq 0$). Compute the Newton polygon of the implicit equation of $\mathcal{C}$ in the following cases:

   - f_1, f_2 are generic (which implies that p_1, p_2, q_1, q_2 don't have pairwise any complex common root)
   - $f_1(u,1), f_2(u,1)$ are generic polynomials of respective degrees d_1, d_2.
   - $f_1(u,1), f_2(u,1)$ are generic *Laurent* polynomials.

5. Let $A = \{a_1., \dots, a_n\}$ be a finite subset of $\mathbb{Z}^d$ and let X_A be the corresponding projective toric variety, that is, the closure of the points of the form $\varphi_A(t) = (t^{a_1} : \cdots : t^{a_n})$, $t = (t_1, \dots, t_d), t_i \neq 0$ for any i, in $(n-1)$-dimensional projective space. Let y be a point in the dual projective space, that is, a hyperplane

$$H_y = \{x \in \mathbb{P}^{n-1} \,/\, \sum_{i=1}^{n} y_i x_i = 0\}.$$

   Asumme that $\tilde{x} \in X_A$ lies in the image of the parametrization, that is $\tilde{x} = \varphi_A(\tilde{t})$ for some $\tilde{t}$ in the d-torus. Let $f_y(t)$ be the polynomial with support A and coefficients y

$$f_y(t) = \sum_{i=1}^{n} y_i t^{a_i}.$$

   Prove that H_y intersects X_A at $\tilde{x}$ containing the tangent space of X_A at the point $\tilde{x}$ if and only if $\tilde{t}$ is a singular point of the hypersurface of the torus $(f_y = 0)$, that is, if and only if f_y and all its partial derivatives vanish at $\tilde{t}$.