$\boldsymbol{A}$-discriminants, suite

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◇ Homogeneous version of the Horn-Kapranov parametrization

- The dual variety X_A^* of the toric variety X_A is the closure of the image of the map $\varphi_A : \mathbb{P}(\ker(A)) \times T^d \to \mathbb{P}^{n-1}(\mathrm{k})^*$ which is given by

$$\varphi_A(\lambda, t) \;=\; (\lambda_1 t^{a_1} : \lambda_2 t^{a_2} : \cdots : \lambda_n t^{a_n}).$$

Moreover, we can linearly parametrize the elements $(\lambda_1, \ldots, \lambda_n)$ in the kernel of A.

A crash course on tropicalisation of algebraic varieties

- For $w \in \mathbb{R}^n$ and $f = \sum_{c \in C} a_c x^c$, $a_c \in (\mathbb{C}^*)^n$, $C \subset \mathbb{Z}^n$, define

$$\mathrm{in}_w f = \sum_{w \cdot c \min} a_c x^c$$ initial polynomial of f,

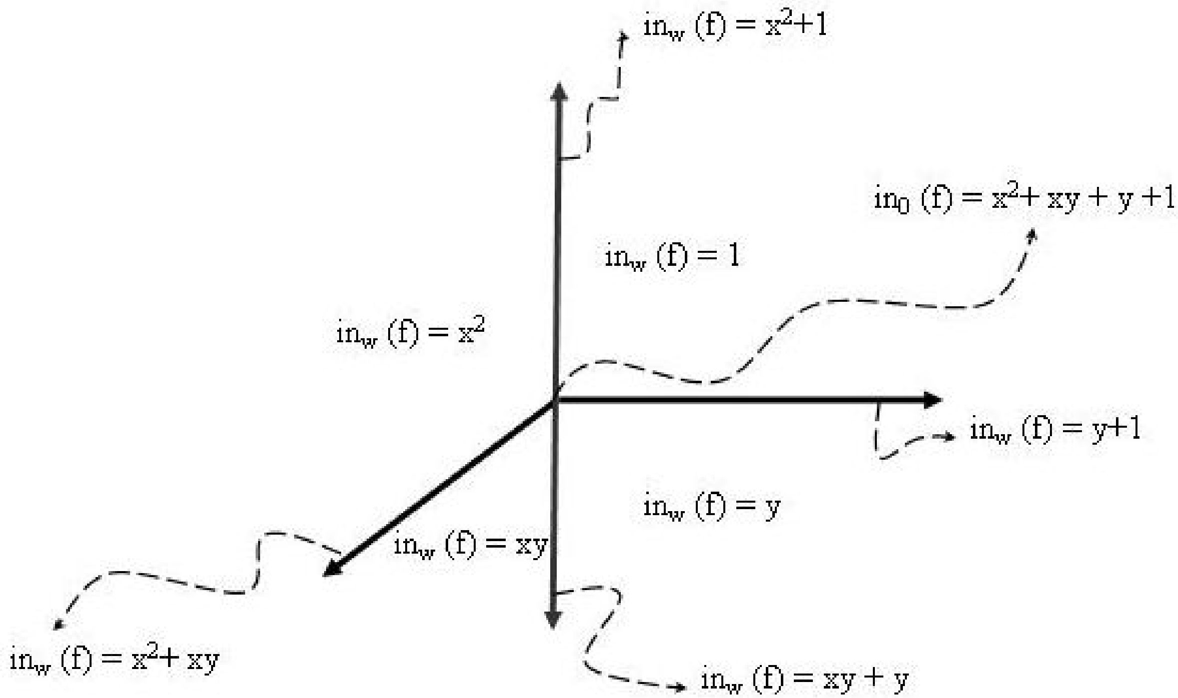
- The *tropicalisation* $\tau(Y)$ of a variety Y in $\mathbb{C}^n$ (or of the ideal $I = I_Y \subset \mathbb{C}[x_1, \dots, x_n]$) is (as a *set*)

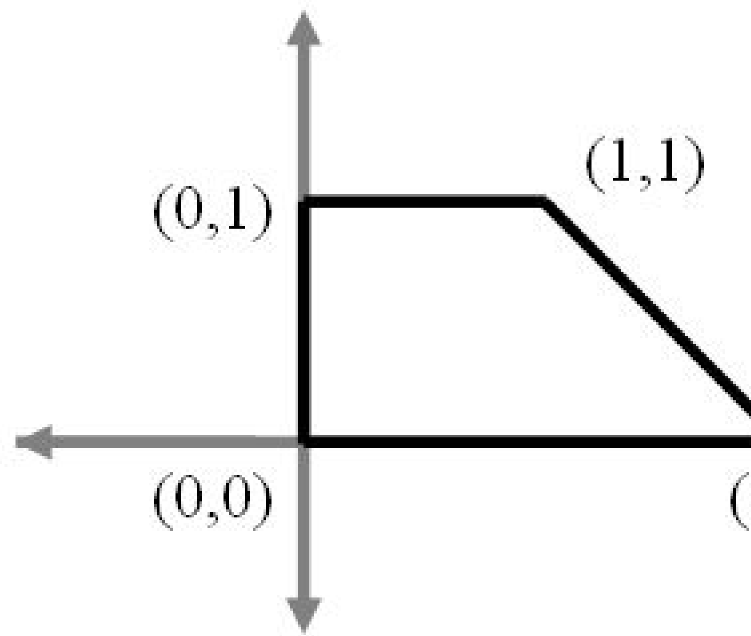
$$\tau(Y) = \tau(I_Y) = \{w \in \mathbb{R}^n \, : \, \mathrm{in}_w(f) \text{ is } \mathbf{not} \text{ a monomial for any } f \in I\}.$$

- In fact, we compute the tropicalisation of a subvariety of a torus $(\mathbb{C}^*)^n$, or the extension of an ideal in $\mathbb{C}[x_1, \dots, x_n]$ to the **Laurent** polynomial ring $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

- There is also **intersection theoretic** information attached to $\tau(Y)$ (Sturmfels-Tevelev '07)

Principal ideals





$w \in \tau(\langle f \rangle) \Leftrightarrow \mathbf{in}_w f$ is not a monomial. It equals the codimension-one skeleton of the Normal fan of the Newton polytope of f.

For non principal ideals in general it is not enough to consider the intersection of the normal cones of the Newton polytopes of a given system of generators, but there exist special ones (**tropical bases**).

Examples

Toric varieties: When the parametrization is monomial $f_i(t) = t^{a_i}$, $i = 1, \ldots, n$, $f_0 = 1$, then $\tau(Y)$ equals the **linear space** spanned by the rows of the $d \times n$ matrix with columns $a_1, \ldots, a_n$.

Linear spaces: Tropicalisations of linear varieties are **complicated** combinatorial objects, depending on the **matroid** = **"pattern of zero and non zero minors"** of the coefficient matrix of the linear polynomials f_i.

Sparse discriminants: Discriminants associated with families of polynomials with fixed exponents $A = \{a_1, \ldots, a_n\}$ (more generally, when f_i can be factorized as **product of linear forms**): can determine combinatorially dimension and degree of dual varieties to toric varieties D.,Feichter, Sturmfels, 2007

The field of Puiseux series

An algebraically closed field with a non trivial valuation

- $\mathbb{K} = \boldsymbol{k}\{\{\varepsilon\}\}$ field of Puiseux series over an algebraically closed field $\boldsymbol{k}$, is *algebraically closed*.

$$\mathbb{K} = \bigcup_{n \in \mathbb{N}} \boldsymbol{k}((\varepsilon^{1/n})).$$

- $\mathbb{K}$ has the following valuation:

$$\begin{array}{rccccc} \mathbf{val} \ : & \mathbb{K}^* & \longrightarrow & \mathbb{Q} & \hookrightarrow & \mathbb{R}^n \\ & \sum a_q \varepsilon^q & \longmapsto & \inf\{q \,|a_q \neq 0\,\} & & \end{array}$$

- Properties of a valuation:

(1) $\mathbf{val}(a) = \infty$ iff $a = 0$

(2) $\mathbf{val}(ab) = \mathbf{val}(a) + \mathbf{val}(b)$

(3) $\mathbf{val}(a+b) \geqslant \min\{\mathbf{val}(a), \mathbf{val}(b)\}$ for all $a, b \in \mathbb{K}^*$

Bieri-Groves' Theorem

Given an irreducible variety $\boldsymbol{Y}$, $\boldsymbol{\tau(Y)}$ is a pure polyhedral rational complex of dimension $= \mathbf{dim}(\boldsymbol{Y})$. Proved through $\mathbf{valuations}$. Is is a fan if the valuation is trivial.

Theorem 0.1 *Equivalent definitions,* $\boldsymbol{k = \overline{k}}$

- $\boldsymbol{\tau(I) = \{w \in \mathbb{R}^n \;:\; \mathrm{in}_w(f)}$ *is* $\mathbf{not}$ *a monomial for any* $\boldsymbol{f \in I\}}$.
- $\boldsymbol{\tau(I) = \{w \in \mathbb{R}^n \;:\; \mathrm{in}_w(I)}$ *does* $\mathbf{not}$ *contain a monomial*$\boldsymbol{\}}$.
- $\boldsymbol{\tau(I) = \{w \in \mathbb{R}^n \;:\; \mathrm{in}_w(I)}$ $\mathbf{has}$ *a zero in* $\mathbf{(k^*)^n\}}$.
- *Kapranov's theorem [EKL, 2006], Newton-Puiseux, Speyer-Sturmfels, ...:*

$$\boldsymbol{\tau(Y) \;=\tau(I) \;=\; \overline{\{(val(z_1),\ldots,val(z_n)), z \in V_{\mathbb{K}^*}((I))\,\}}\,.}$$

Proof: *Given* $\boldsymbol{w}$ *and a zero* $\gamma \in (\boldsymbol{k}^*)^n$ *of* $\mathbf{in}_w(\boldsymbol{I})$, *construct a root* $\boldsymbol{z}$ *of* $\boldsymbol{I}$ *in* $(\mathbb{K}^*)^n$ *of the form* $\boldsymbol{z} = (\gamma_1\varepsilon^{w_1} + \mathbf{h.o.t.}, \ldots, \gamma_n\varepsilon^{w_n} + \mathbf{h.o.t.})$. *Not as straightforward :-)*

The discriminant of a cubic polynomial in 1 variable

$$A \;:=\; \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \qquad X_A \text{ is the twisted cubic.}$$

$$f_A(x;t) \;=\; x_1 t^0 + x_2 t^1 + x_3 t^2 + x_4 t^3$$

$$\Delta_A \;=\; 27\,x_1^2 x_4^2 - 18\,x_1 x_2 x_3 x_4 + 4\,x_1 x_3^3 + 4\,x_2^3 x_4 - x_2^2 x_3^2$$

$$\mathrm{in}_{(-1,-1,-1,0)}(\Delta_A) \;=\; 4x_1 x_3^3 \;-\; x_2^2 x_3^2 \qquad (-1,-1,-1,0) \in \tau(X_A^*)$$

$$\mathrm{in}_{(1,0,1,0)}(\Delta_{)} \;=\; 4x_2^3 x_4 \qquad (1,0,1,0) \not\in \tau(X_A^*)$$

The discriminant of a cubic polynomial in 1 variable

$$\Delta_A \,=\, 27\,x_1^2x_4^2 - 18\,x_1x_2x_3x_4 + 4\,x_1x_3^3 + 4\,x_2^3x_4 - x_2^2x_3^2$$

The Newton polytope $N(\Delta_A)$ is the convex hull of the exponent vectors $\alpha_1 = (2,0,0,2), \alpha_2 = (1,0,3,0), \alpha_3 = (0,3,0,1)$, $\alpha_4 = (0,2,2,0), \alpha_5 = (1,1,1,1)$, in $\mathbb{R}^4$.

As the discriminant has two homogeneities read in the rows of the matrix A: the linear functions

$$\langle (1,1,1,1), \alpha_i \rangle, \quad \langle (0,1,2,3), \alpha_i \rangle$$

take the same values (4 and 6 respectively) for any $i = 1, \ldots, 5$, $N(\Delta_A)$ is a polygon lying in a two dimensional plane in $\mathbb{R}^4$. It is straightforward to check that $\alpha_1, \ldots, \alpha_4$ are vertices and α_5 is an interior point.

The results in [D. - Feichtner - Sturmfels, '07] allow to predict the monomials in Δ_A, even if we couldn't compute it (as it happens in the general case).

$$\tau(\Delta_A) \,=\, \left(\cup_{i=1}^4 \mathbb{R}_{\geq 0} e_i\right) + \langle (1,1,1,1), (0,1,2,3) \rangle,$$

(and all the multiplicities are equal to 1).

To visualize it, we mod out by the row space of A to have a two-dimensional representation.

Figure 1: Newton polygon, tropicalization and extreme monomials of the discriminant of a degree $\mathbf{3}$ polynomial: $x_2^3x_4$, $x_2^2x_3^2$, $x_1x_3^3$, $x_1^2x_4^2$

The discriminant of a cubic polynomial in 1 variable

So we choose a basis $\{(1,-2,1,0),\,(0,1,-2,1)\}$ of the kernel of A and we project $\pi : \tau(\Delta_A) \to \mathbb{R}^2$, so that $b_1 := \pi(e_1) = (1,0), b_2 := \pi(e_2) = (-2,1)$, $b_3 := \pi(e_3) = (1,-2), b_4 := \pi(e_4) = (0,1)$, and the image if the tropicalization is the union of the four positive rays generated by these vectors.

To compute a monomial in Δ_A we pick a point $w \notin \pi(\tau(\Delta_A))$, for instance a point in the interior of the positive cone $\mathcal{C}$ generated by b_2 and b_3. We now "place" the projection $\pi(\tau(\Delta_A))$at w and we see which rays emanating from there intersect $\pi(\tau(\Delta_A))$. The intersections are given by the point $(w+\mathbb{R}_{\geq 0}b_1)\cap\mathbb{R}_{\geq 0}b_3$, with $|\det(b_1,b_3)| = 2$ and by the point $(w+\mathbb{R}_{\geq 0}b_4)\cap\mathbb{R}_{\geq 0}b_2$, with $|\det(b_2,b_4)| = 2$. Therefore, the vertex of $N(\Delta_A)$ dual to $\mathcal{C}$ is the point $(2,0,0,2)$.

First theorem from [DFS'07]

- $\mathcal{L}(A)$ = geometric lattice whose elements are the sets of zero-entries of the vectors in $\mathbf{kernel}(A)$, ordered by inclusion.
- $\mathcal{C}(A)$ = set of proper maximal chains in $\mathcal{L}(A)$.
- We represent these chains as $(n-d-1)$-element subsets $\sigma = \{\sigma_1, \ldots, \sigma_{n-d-1}\}$ of $\{0,1\}^n$.
- The tropicalization of the kernel of A equals

$$\tau(\mathbf{kernel}(A)) \quad = \quad \bigcup_{\sigma \in \mathcal{C}(A)} \mathbb{R}_{\geq 0}\, \sigma \, .$$

- This tropical linear space is a subset of $\mathbb{R}^n$.
- Theorem: The tropical A-discriminant is the Minkowski sum of this tropical linear space and the (classical) row space of the $d \times n$-matrix A.

This is the tropical version of the Horn-Kapranov parametrization.

Second theorem from [DFS'07]

Let X be a subvariety of a torus and I_X its ideal. If w is any regular point in the tropical variety $\tau(X)$ then the multiplicity m_w of the top dimensional cone containing w equals the sum of multiplicities of all minimal associate primes of the initial ideal $\text{in}_w(I_X)$.

In particular: If $Y = (H = 0)$ is a hypersurface, the multiplicity of a max dim cone in $\tau(Y)$ equals the lattice lenght of the corresponding orthogonal edge of the Newton polytope of H.

$\mathbf{deg}(Y)$ **from** $\tau(Y)$: For simplicity, assume $Y = (H = 0)$ is a hypersurface. It is enough to describe the extreme monomials of H, i.e. the monomials corresponding to a vertex of $Q = N(H)$. If H is homogeneous, a single extreme monomial is enough ...

$\mathbf{deg}(\boldsymbol{Y})$ from $\boldsymbol{\tau}(\boldsymbol{Y})$ - D., Feichther, Sturmfels, 2007, Th. 2.2

- Vertices of $\boldsymbol{Q} = \boldsymbol{N}(\boldsymbol{H})$ correspond bijectively to connected components of $\mathbb{R}^{\boldsymbol{n}} \backslash \boldsymbol{\tau}(\boldsymbol{Y})$.
- Let $\boldsymbol{u} \notin \boldsymbol{\tau}(\boldsymbol{Y})$. To compute the corresponding vertex of $\boldsymbol{Q}$, which is $\mathbf{face}_{\boldsymbol{u}}(\boldsymbol{Q})$:
- Consider the halflines $\boldsymbol{L}_{\boldsymbol{i}} = \boldsymbol{u} + \mathbb{R}_{\geq 0}\boldsymbol{e}_{\boldsymbol{i}}$.
- $\boldsymbol{L}_{\boldsymbol{i}}$ intersects $\boldsymbol{\tau}(\boldsymbol{Y})$ in finitely many points $\boldsymbol{w}$, and with each of these we associate the nonnegative integer

$$\left[\mathbb{Z}^{\boldsymbol{n}} : \mathbb{Z}\boldsymbol{e}_{\boldsymbol{i}} + (\boldsymbol{L}_{\boldsymbol{w}} \cap \mathbb{Z}^{\boldsymbol{n}})\right] \cdot \boldsymbol{m}_{\boldsymbol{w}}.$$

- The $\boldsymbol{i}$-th coordinate of the vertex is the sum of these integers, for all $\boldsymbol{v} \in \boldsymbol{L}_{\boldsymbol{i}} \cap \boldsymbol{\tau}(\boldsymbol{Y})$.